

THE KATO-PONCE INEQUALITY

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ABSTRACT. In this article we revisit the inequalities of Kato and Ponce concerning the L^r norm of the Bessel potential $J^s = (1 - \Delta)^{s/2}$ (or Riesz potential $D^s = (-\Delta)^{s/2}$) of the product of two functions in terms of the product of the L^p norm of one function and the L^q norm of the the Bessel potential J^s (resp. Riesz potential D^s) of the other function. Here the indices p, q , and r are related as in Hölder's inequality $1/p + 1/q = 1/r$ and they satisfy $1 \leq p, q \leq \infty$ and $1/2 \leq r < \infty$ and $s > \frac{n}{r} - n$. Also the estimate is of weak-type when either p or q is equal to 1. In the case $r < 1$ we indicate via an example that when $s \leq \frac{n}{r} - n$ the inequality fails. Furthermore, we extend these results to the multi-parameter case.

1. INTRODUCTION

In [15], Kato and Ponce obtained the commutator estimate

$$\|J^s(fg) - f(J^s g)\|_{L^p(\mathbf{R}^n)} \leq C \left[\|\nabla f\|_{L^\infty(\mathbf{R}^n)} \|J^{s-1} g\|_{L^p(\mathbf{R}^n)} + \|J^s f\|_{L^p(\mathbf{R}^n)} \|g\|_{L^\infty(\mathbf{R}^n)} \right]$$

for $1 < p < \infty$ and $s > 0$, where $J^s := (1 - \Delta)^{s/2}$ is the Bessel potential, ∇ is the n -dimensional gradient, f, g are Schwartz functions, and C is a constant depending on n, p and s . This estimate was motivated by a question stated in Remark 4:1 in Kato's work [14]. The proof in [15] is based on the Coifman-Meyer multiplier theorem [6] and Stein's complex interpolation theorem for analytic families [22].

Using the Riesz potential $D^s := (-\Delta)^{s/2}$ instead, Kenig, Ponce, Vega [16] obtained a closely related estimate,

$$\|D^s[f g] - f D^s g - g D^s f\|_{L^r} \leq C(s_1, s_2, p, q) \|D^{s_1} f\|_{L^p} \|D^{s_2} g\|_{L^q}$$

where $s = s_1 + s_2$ for $s, s_1, s_2 \in (0, 1)$, and $1 < p, q, r < \infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

In place of the original statement given by Kato and Ponce, the following variant is known in the literature as the Kato-Ponce inequality (also *fractional Leibniz rule*)

$$(1) \quad \|J^s(fg)\|_{L^r(\mathbf{R}^n)} \leq C \left[\|f\|_{L^{p_1}(\mathbf{R}^n)} \|J^s g\|_{L^{q_1}(\mathbf{R}^n)} + \|J^s f\|_{L^{p_2}(\mathbf{R}^n)} \|g\|_{L^{q_2}(\mathbf{R}^n)} \right]$$

where $s > 0$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ for $1 < r < \infty$, $1 < p_1, q_2 \leq \infty$, $1 < p_2, q_1 < \infty$ and $C = C(s, n, p_1, p_2, q_1, q_2)$. Note that in this formulation the L^∞ norm does not fall on the terms with the Bessel potential. The preceding estimate is proved in a similar manner [11, 13], using the Coifman-Meyer multiplier theorem in conjunction with Stein's complex interpolation. The homogeneous version of (1), where J^s is

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replaced by D^s , is proved by the same approach and is also called the Kato-Ponce inequality. Other known proofs of (1) involve applications of the Hardy-Littlewood vector-valued maximal function inequality [8], Mihlin theorem [18] or vector-valued Calderon-Zygmund theorem [1], but these methods naturally do not extend to $r < 1$. The approach in this work does not only provide a new proof in the well-studied case $r \geq 1$, but also has a natural extension to $r < 1$.

There are further generalizations of Kato-Ponce-type inequalities. For instance, Muscalu, Pipher, Tao, and Thiele, [17] extended this inequality to allow for partial fractional derivatives in \mathbf{R}^2 . Bernicot, Maldonado, Moen, and Naibo [3] proved the Kato-Ponce inequality in weighted Lebesgue spaces under certain restrictions on the weights. The last authors also extended the Kato-Ponce inequality to indices $r < 1$ under the assumption $s > n$.

In this article, we prove the Kato-Ponce inequality for $1/2 \leq r < \infty$, where Lebesgue spaces L^r are replaced by weak type Lorentz spaces $L^{r,\infty}$ when either p_j or q_j on the right hand side equals 1. Furthermore we establish that when $r > 1$ the L^∞ norm can be applied on either term on the right in (1) including the term with the Bessel/Riesz potential. The inclusion of L^∞ in our estimates is based on ideas stemming from the recent work of Bae and Biswas [1] and may have been overlooked by other authors. We note that the approach in [15] could not incorporate the proof of (1) when the L^∞ norm is placed on the terms involving Bessel potential, since in this case, the complex interpolation fails because the family of symbols $\{J^{it}\}_{t \in \mathbf{R}}$ is unbounded on L^∞ .

When $r \geq 1$ inequality (1) is valid for all $s \geq 0$ but when $r < 1$ there is a restriction $s > n/r - n$. Moreover, we show via an example that inequality (1) fails for $s \leq n/r - n$ indicating the sharpness of the restriction. Our extension improves that of Bernicot, Maldonado, Moen and Naibo in [3] which requires $s > n$ for $r < 1$. Additionally, we generalize the multi-parameter extension of the Kato-Ponce inequality by Muscalu, Pipher, Tao, and Thiele [17] to allow for partial fractional derivatives in \mathbf{R}^n .

We now state two of our main results in this work. Additional results are obtained in Section 5.

Theorem 1. *Let $\frac{1}{2} < r < \infty$, $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Given $s > \max(0, \frac{n}{r} - n)$ or $s \in 2\mathbf{N}$, there exists $C = C(n, s, p_1, q_1, p_2, q_2) < \infty$ such that for all $f, g \in \mathcal{S}(\mathbf{R}^n)$ we have*

$$(2) \quad \|D^s(fg)\|_{L^r(\mathbf{R}^n)} \leq C \left[\|D^s f\|_{L^{p_1}(\mathbf{R}^n)} \|g\|_{L^{q_1}(\mathbf{R}^n)} + \|f\|_{L^{p_2}(\mathbf{R}^n)} \|D^s g\|_{L^{q_2}(\mathbf{R}^n)} \right],$$

$$(3) \quad \|J^s(fg)\|_{L^r(\mathbf{R}^n)} \leq C \left[\|f\|_{L^{p_1}(\mathbf{R}^n)} \|J^s g\|_{L^{q_1}(\mathbf{R}^n)} + \|J^s f\|_{L^{p_2}(\mathbf{R}^n)} \|g\|_{L^{q_2}(\mathbf{R}^n)} \right]$$

Moreover for $r < 1$, if one of the indices p_1, p_2, q_1, q_2 is equal to 1, then (2) and (3) hold when the $L^r(\mathbf{R}^n)$ norms on the left hand side of the inequalities are replaced by the $L^{r,\infty}(\mathbf{R}^n)$ quasi-norm.

We remark that the statement above does not include the endpoint $L^1 \times L^\infty \rightarrow L^{1,\infty}$. Next, we have a companion theorem that focuses on negative results which highlight the sharpness of Theorem 1.

Theorem 2. *Let $s \leq \max(\frac{n}{r} - n, 0)$ and $s \notin 2\mathbf{N} \cup \{0\}$. Then both (2) and (3) fail for any $1 < p_1, q_1, p_2, q_2 < \infty$.*

Upon completion of this manuscript, the authors discovered that Muscalu and Schlag [19] had independently reached conclusion (2) via a different approach, based on discretized paraproducts. A version of Lemma 1 also appears in their text, but otherwise their approach is different from ours.

The paper is organized as follows. In Section 2, we introduce Littlewood-Paley operators and prove an estimate concerning them. In Section 3, we prove Theorems 1 and 2 for the homogeneous version of inequality (2). In Section 4, we prove Theorems 1 and 2 for the inhomogeneous version of inequality (3). In Section 5, we state and prove a multi-parameter generalization of Theorems 1 and 2.

2. PRELIMINARIES

We denote by $\langle x, y \rangle$ the inner product in \mathbf{R}^n . We use the notation $\Psi_t(x) = t^{-n}\Psi(x/t)$ when $t > 0$ and $x \in \mathbf{R}^n$. We denote by $\mathcal{S}(\mathbf{R}^n)$ the space of all rapidly decreasing **smooth** functions on \mathbf{R}^n called Schwartz functions. We denote by

$$\widehat{f}(\xi) = \mathcal{F}(f)(x) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i\langle x, \xi \rangle} dx$$

the Fourier transform of a Schwartz function f on \mathbf{R}^n . We also denote by $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$ the inverse Fourier transform. We recall the classical multiplier result of Coifman and Meyer [4]; see also [5]:

Theorem A (Coifman-Meyer). *Let $m \in L^\infty(\mathbf{R}^{2n})$ be smooth away from the origin. Suppose that there exists a constant $A > 0$ satisfying*

$$|\partial_\xi^\alpha \partial_\eta^\beta m|(\xi, \eta) \leq A (|\xi| + |\eta|)^{-|\alpha| - |\beta|}$$

for all $\xi, \eta \in \mathbf{R}^n$ with $|\xi| + |\eta| \neq 0$ and $\alpha, \beta \in \mathbf{Z}^n$ multi-indices with $|\alpha|, |\beta| \leq 2n + 1$. Then for all $f, g \in \mathcal{S}(\mathbf{R}^n)$,

$$\left\| \int_{\mathbf{R}^{2n}} m(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{i\langle \xi + \eta, \cdot \rangle} d\xi d\eta \right\|_{L^r(\mathbf{R}^n)} \leq C(p, q, r, A) \|f\|_{L^p(\mathbf{R}^n)} \|g\|_{L^q(\mathbf{R}^n)}$$

where $\frac{1}{2} < r < \infty$, $1 < p, q \leq \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Furthermore, when either p or q is equal to 1, then the $L^r(\mathbf{R}^n)$ norm on left hand side can be replaced by the $L^{r, \infty}(\mathbf{R}^n)$ norm.

We recall the following version of the Littlewood-Paley theorem.

Theorem 3. *Suppose that $\widehat{\Psi}$ is an integrable function on \mathbf{R}^n that satisfies*

$$(4) \quad \sum_{j \in \mathbf{Z}} |\Psi(2^{-j}\xi)|^2 \leq B^2$$

and

$$(5) \quad \sup_{y \in \mathbf{R}^n \setminus \{0\}} \sum_{j \in \mathbf{Z}} \int_{|x| \geq 2|y|} |\widehat{\Psi}_{2^{-j}}(x-y) - \widehat{\Psi}_{2^{-j}}(x)| dx \leq B$$

Then there exists a constant $C_n < \infty$ such that for all $1 < p < \infty$ and all f in $L^p(\mathbf{R}^n)$,

$$(6) \quad \left\| \left(\sum_{j \in \mathbf{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \leq C_n B \max(p, (p-1)^{-1}) \|f\|_{L^p(\mathbf{R}^n)}$$

where $\Delta_j f := \widehat{\Psi}_{2^{-j}} * f$. There also exists a $C'_n < \infty$ such that for all f in $L^1(\mathbf{R}^n)$,

$$(7) \quad \left\| \left(\sum_{j \in \mathbf{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{1,\infty}(\mathbf{R}^n)} \leq C'_n B \|f\|_{L^1(\mathbf{R}^n)}.$$

Proof. We make a few remarks about the proof. Clearly the required estimate holds when $p = 2$ in view of (4). To obtain estimate (7) and thus the case $p \neq 2$, we define an operator \vec{T} acting on functions on \mathbf{R}^n as follows:

$$\vec{T}(f)(x) = \{\Delta_j(f)(x)\}_j.$$

The inequalities (6) and (7) we wish to prove say simply that \vec{T} is a bounded operator from $L^p(\mathbf{R}^n, \mathbf{C})$ to $L^p(\mathbf{R}^n, \ell^2)$ and from $L^1(\mathbf{R}^n, \mathbf{C})$ to $L^{1,\infty}(\mathbf{R}^n, \ell^2)$. We indicated that this statement is true when $p = 2$, and therefore the first hypothesis of Theorem 4.6.1 in [10] is satisfied. We now observe that the operator \vec{T} can be written in the form

$$\vec{T}(f)(x) = \left\{ \int_{\mathbf{R}^n} \widehat{\Psi}_{2^{-j}}(x-y) f(y) dy \right\}_j = \int_{\mathbf{R}^n} \vec{K}(x-y)(f(y)) dy,$$

where for each $x \in \mathbf{R}^n$, $\vec{K}(x)$ is a bounded linear operator from \mathbf{C} to ℓ^2 given by

$$(8) \quad \vec{K}(x)(a) = \{\widehat{\Psi}_{2^{-j}}(x)a\}_j.$$

We clearly have that $\|\vec{K}(x)\|_{\mathbf{C} \rightarrow \ell^2} = (\sum_j |\widehat{\Psi}_{2^{-j}}(x)|^2)^{\frac{1}{2}}$, and to be able to apply Theorem 4.6.1 in [10] we need to know that

$$(9) \quad \int_{|x| \geq 2|y|} \|\vec{K}(x-y) - \vec{K}(x)\|_{\mathbf{C} \rightarrow \ell^2} dx \leq C_n B, \quad y \neq 0.$$

We clearly have

$$\begin{aligned} \|\vec{K}(x-y) - \vec{K}(x)\|_{\mathbf{C} \rightarrow \ell^2} &= \left(\sum_{j \in \mathbf{Z}} |\widehat{\Psi}_{2^{-j}}(x-y) - \widehat{\Psi}_{2^{-j}}(x)|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{j \in \mathbf{Z}} |\widehat{\Psi}_{2^{-j}}(x-y) - \widehat{\Psi}_{2^{-j}}(x)| \end{aligned}$$

and so condition (5) implies (9). \square

Corollary 1. Let $m \in \mathbf{Z}^n \setminus \{0\}$ and $\widehat{\Psi}(x) = \widehat{\psi}(x+m)$ for some Schwartz function ψ supported in the annulus $1/2 \leq |\xi| \leq 2$. Let $\Delta_j(f) = (f \widehat{\Psi})^\vee$. Then for all $1 < p < \infty$,

$$(10) \quad \left\| \left(\sum_{j \in \mathbf{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \leq C_n \ln(1 + |m|) \max(p, (p-1)^{-1}) \|f\|_{L^p(\mathbf{R}^n)}.$$

There also exists $C_n < \infty$ such that for all $f \in L^1(\mathbf{R}^n)$,

$$(11) \quad \left\| \left(\sum_{j \in \mathbf{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{1,\infty}(\mathbf{R}^n)} \leq C_n \ln(1 + |m|) \|f\|_{L^1(\mathbf{R}^n)}.$$

Proof. Note

$$\Psi(\xi) = \psi(\xi) e^{2\pi i \langle m, \xi \rangle}.$$

The fact that ψ is supported in the annulus $1/2 \leq |\xi| \leq 2$ implies condition (4) for Ψ . We now focus on condition (5) for $\widehat{\Psi}$.

We fix a nonzero y in \mathbf{R}^n and $j \in \mathbf{Z}$. We look at

$$\int_{|x| \geq 2|y|} |\widehat{\Psi}_{2^{-j}}(x-y) - \widehat{\Psi}_{2^{-j}}(x)| dx = \int_{|x| \geq 2|y|} 2^{jn} |\widehat{\psi}(2^j x - 2^j y + m) - \widehat{\psi}(2^j x + m)| dx$$

Changing variables we can write the above as

$$I_j = \int_{|x| \geq 2|y|} |\widehat{\Psi}_{2^{-j}}(x-y) - \widehat{\Psi}_{2^{-j}}(x)| dx = \int_{|x-m| \geq 2^{j+1}|y|} |\widehat{\psi}(x-2^j y) - \widehat{\psi}(x)| dx$$

Case 1: $2^j \geq 2|m||y|^{-1}$. In this case we estimate I_j by

$$\begin{aligned} & \int_{|x-m| \geq 2^{j+1}|y|} \frac{c}{(1+|x-2^j y|)^{n+2}} dx + \int_{|x-m| \geq 2^{j+1}|y|} \frac{c}{(1+|x|)^{n+2}} dx \\ &= \int_{|x+2^j y-m| \geq 2^{j+1}|y|} \frac{c}{(1+|x|)^{n+2}} dx + \int_{|x-m| \geq 2^{j+1}|y|} \frac{c}{(1+|x|)^{n+2}} dx \end{aligned}$$

Suppose that x lies in the domain of integration of the first integral. Then

$$|x| \geq |x+2^j y-m| - 2^j|y| - |m| \geq 2^{j+1}|y| - 2^j|y| - \frac{1}{2} 2^j|y| = \frac{1}{2} 2^j|y|.$$

If x lies in the domain of integration of the second integral, then

$$|x| \geq |x-m| - |m| \geq 2^{j+1}|y| - |m| \geq 2^{j+1}|y| - \frac{1}{2} 2^j|y| = \frac{3}{2} 2^j|y|.$$

In both cases we have

$$I_j \leq 2 \int_{|x| \geq \frac{1}{2} 2^j|y|} \frac{c}{(1+|x|)^{n+2}} dx \leq \frac{C}{2^j|y|} \int_{\mathbf{R}^n} \frac{1}{(1+|x|)^{n+1}} dx \leq \frac{C_n}{2^j|y|},$$

and clearly

$$\sum_{j: 2^j|y| \geq 2|m|} I_j \leq \sum_{j: 2^j|y| \geq 2} I_j \leq C_n.$$

Case 2: $|y|^{-1} \leq 2^j \leq 2|m||y|^{-1}$. The number of j 's in this case are $O(\ln|m|)$. Thus, uniformly bounding I_j by a constant, we obtain

$$\sum_{j: 1 \leq 2^j|y| \leq 2|m|} I_j \leq C_n(1 + \ln|m|).$$

Case 3. $2^j \leq |y|^{-1}$. In this case we have

$$|\widehat{\psi}(x-2^j y) - \widehat{\psi}(x)| = \left| \int_0^1 2^j \langle \nabla \widehat{\psi}(x-2^j t y), y \rangle dt \right| \leq 2^j|y| \int_0^1 \frac{c}{(1+|x-2^j t y|)^{n+1}} dt.$$

Integrating over $x \in \mathbf{R}^n$ gives the bound $I_j \leq C_n 2^j |y|$. Thus, we obtain

$$\sum_{j:2^j|y|\leq 1} I_j \leq C_n.$$

Overall, we obtain the bound $C_n \ln(1 + |m|)$ for (5), which yields the desired statement by Theorem 3. \square

3. HOMOGENEOUS KATO-PONCE INEQUALITY

In the following lemma, we recall the explicit formula for the Riesz potential described in [10].

Lemma 1. *Let $f \in \mathcal{S}(\mathbf{R}^n)$ be fixed. Given $s > 0$, define $f_s := D^s f$. Then f_s lies in $L^\infty(\mathbf{R}^n)$ and satisfies the following asymptotic estimate:*

- *There exists a constant $C(n, s, f)$ such that*

$$(12) \quad |f_s(x)| \leq C(n, s, f) |x|^{-n-s} \quad \forall x : |x| > 1.$$

- *Let $s \notin 2\mathbf{N}$. If $f(x) \geq 0$ for $\forall x \in \mathbf{R}^n$ and $f \not\equiv 0$, then there exists $R \gg 1$ and a constant $C(n, s, f, R)$ such that*

$$(13) \quad |f_s(x)| \geq C(n, s, f, R) |x|^{-n-s} \quad \forall x : |x| > R.$$

Proof. For any $z \in \mathbf{C}$ with $\operatorname{Re} z > -n$ and $g \in \mathcal{S}(\mathbf{R}^n)$, define the distribution u_z by

$$(14) \quad \langle u_z, g \rangle := \int_{\mathbf{R}^n} \frac{\pi^{\frac{z+n}{2}}}{\Gamma\left(\frac{z+n}{2}\right)} |x|^z g(x) dx,$$

where $\Gamma(\cdot)$ denotes the gamma function. We recall Theorem 2.4.6 of [10] and the preceding remarks:

- For any $g \in \mathcal{S}(\mathbf{R}^n)$, the map $z \mapsto \langle u_z, g \rangle$ on the half-plane $\operatorname{Re} z > -n$ has an holomorphic extension to the entire complex plane.
- $\langle u_z, \widehat{g} \rangle = \langle u_{-n-z}, \widehat{g} \rangle$, where $\langle u_z, f \rangle$ is understood as the holomorphic extension when $\operatorname{Re} z \leq -n$.
- Both $u_z, u_{-n-z} \in L^1_{\text{loc}}$ if and only if $-n < \operatorname{Re} z < 0$, in which case both u_z and \widehat{u}_z are well-defined by (14).

Now, fix $f \in \mathcal{S}(\mathbf{R}^n)$ note that for $s > 0$,

$$f_s(x) = \int_{\mathbf{R}^n} |\xi|^s \widehat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi = \frac{\Gamma\left(\frac{s+n}{2}\right)}{\pi^{\frac{s+n}{2}}} \left\langle u_s, \widehat{f}(\cdot) e^{2\pi i \langle \cdot, x \rangle} \right\rangle.$$

Furthermore, note that the constant $\Gamma\left(\frac{s+n}{2}\right) / \pi^{\frac{s+n}{2}} \neq 0$ when $s > 0$. Thus it suffices prove the estimates (12) and (13) for $\left\langle u_s, \widehat{f}(\cdot) e^{2\pi i \langle \cdot, x \rangle} \right\rangle$. We extend the map $s \mapsto \left\langle u_s, \widehat{f}(\cdot) e^{2\pi i \langle \cdot, x \rangle} \right\rangle$ to an entire function and replace $s \in \mathbf{R}_+$ by $z \in \mathbf{C}$.

Applying Theorem 2.4.6 of [10], we obtain

$$\left\langle u_z, \widehat{f}(\cdot) e^{2\pi i \langle \cdot, x \rangle} \right\rangle = \langle u_{-n-z}, f(\cdot + x) \rangle.$$

For $z : -n < \operatorname{Re} z < 0$, we can use (14) to write

$$\langle u_{-n-z}, f(\cdot + x) \rangle = \int_{\mathbf{R}^n} \frac{\pi^{-\frac{z}{2}}}{\Gamma(-\frac{z}{2})} |y|^{-n-z} f(x+y) dy.$$

We split integral in the right hand side into $\int_{|y| \leq 1} \cdot dy + \int_{|y| > 1} \cdot dy =: I_1(x, z) + I_2(x, z)$.

First, we recall the expression (2.4.7) in [10] which shows that $I_1(x, z)$ can be extended to an entire function in $z \in \mathbf{C}$ so that for any z with $\operatorname{Re} z < N$, for some $N \in \mathbf{N}$, $I_1(x, z)$ can be computed via the following formula:

$$\begin{aligned} I_1(x, z) &= \sum_{|\alpha| \leq N} b(n, \alpha, z) \partial^\alpha \langle \partial^\alpha \delta_0, f(\cdot + x) \rangle \\ &+ \int_{|y| < 1} \frac{\pi^{-\frac{z}{2}}}{\Gamma(-\frac{z}{2})} \left\{ f(x+y) - \sum_{|\alpha| \leq N} \frac{(\partial^\alpha f)(x)}{\alpha!} y^\alpha \right\} |y|^{-n-z} dy, \end{aligned}$$

where $\alpha \in \mathbf{Z}_+^n$ is a multi-index and $b(n, \alpha, z)$ is an entire function for any given n, α . From this formula, we remark that for a fixed $z : 0 < \operatorname{Re} z < N$, there exists $C(z, n, N)$ such that

$$|I_1(x, z)| \leq C(z, n, N) \left(\sum_{|\alpha| \leq N} |\partial^\alpha f|(x) + \sup_{|y| \leq 1} \sum_{|\beta| = N+1} \sup |\partial^\beta f|(x+y) \right).$$

Note that $I_1(x, z)$ decays like a Schwartz function for any fixed $z \in \mathbf{R}_+$.

Now we consider $I_2(x, z)$, which is also an entire function in z . For z satisfying $-n < \operatorname{Re} z < 0$, this entire function is given by

$$(15) \quad I_2(x, z) = \int_{|x-y| > 1} \frac{\pi^{-\frac{z}{2}}}{\Gamma(-\frac{z}{2})} |x-y|^{-n-z} f(y) dy.$$

Note that (15) is valid for any $z \in \mathbf{C}$, so this gives an exact expression for $I_2(x, z)$. It is important to notice that the constant $C_z := \pi^{-\frac{z}{2}}/\Gamma(-\frac{z}{2})$ vanishes when z is a positive even integer because of the poles of $\Gamma(\cdot)$. However, if $z \notin 2\mathbf{Z}_+$, then $C_z \neq 0$.

Let $z \in \mathbf{R}_+ \setminus 2\mathbf{N}$. It is easily seen from (15) that, given $z \in \mathbf{R}_+$, $I_2(x, z)$ is bounded for all $x \in \mathbf{R}^n$. Now we consider the decay rate of $I_2(x, z)$ for $|x| > 2$.

Split the integral in (15) into two regions: $I_2^1 := \int_{|x-y| > 1, |x| \leq 2|y|} \cdot dy$ and $I_2^2 := \int_{|x-y| > 1, |x| > 2|y|} \cdot dy$. For the first integral, there is some constant $C(z, K) > 0$ satisfying

$$|I_2^1(x, z)| \leq |C_z| \int_{|y| \geq |x|/2} |f(y)| dy = \frac{|C_z|}{(1 + |x|/2)^K} \int_{\mathbf{R}^n} (1 + |y|)^K |f(y)| dy \leq \frac{C(z, K)}{(1 + |x|)^K}$$

for any $K \in \mathbf{N}$. Thus, over this region, $I_2(\cdot, z)$ decays like a Schwartz function. For the remaining integral, we can drop the condition $|x-y| > 1$ and write

$$I_2^2(x, z) = C_z \int_{|x| > 2|y|} |x-y|^{-n-z} f(y) dy$$

Note that $\frac{1}{2}|x| < |x-y| < \frac{3}{2}|x|$ whenever $|x| > 2|y|$. Thus, $I_2^2(x, z) \leq C_{n,z} 1/|x|^{n+z}$ for $|x| > 2$. This proves (12).

Moreover, if $f(y) \geq 0$ for $\forall y \in \mathbf{R}^n$ but $f \not\equiv 0$, we have

$$|I_2^2(x, z)| \geq \left(\frac{3}{2}\right)^{n+z} \frac{|C_z|}{|x|^{n+z}} \int_{|x|>2|y|} f(y) dy.$$

Taking $|x|$ large enough so that $f \not\equiv 0$ on the ball $B_{|x|/2} := \{y \in \mathbf{R}^n : 2|y| < |x|\}$, the integral above is bounded from below. This proves (13). \square

Proof of Theorem 2 for the homogeneous case. Note that (13) states that for all s satisfying $s \neq 2k$ for some $k \in \mathbf{N}$ and $0 < s \leq \frac{n}{r} - n$, $D^s f \notin L^r(\mathbf{R}^n)$. In particular, we can choose any non-zero function $f \in \mathcal{S}(\mathbf{R}^n)$ so that $D^s |f|^2 \notin L^r(\mathbf{R}^n)$. On the other hand, (12) tells us that $D^s f \in L^p(\mathbf{R}^n)$ for any $s > 0$ and $p \geq 1$. This disproves inequality (2) when $0 < s \leq \frac{n}{r} - n$, hence proves Theorem 2 in this case. We remark also that if $s > \frac{n}{r} - n$, then $D^s f \in L^r(\mathbf{R}^n)$ for any $f \in \mathcal{S}(\mathbf{R}^n)$.

Now consider the case $\frac{n}{r} - n < s < 0$. Let $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^n)$ be real-valued radial functions where Φ is supported on a ball of radius 1, and $\Psi \equiv 1$ on $\{\xi \in \mathbf{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and is supported on a larger annulus. In (2), let $f(x) = f_k(x) := e^{2\pi i 2^k \langle e_1, x \rangle} \widehat{\Phi}(x)$ and $g(x) = g_k(x) := e^{-2\pi i 2^k \langle e_1, x \rangle} \widehat{\Phi}(x)$ with $|k| \gg 1$. Then the left hand side of (2) is independent of k . On the other hand, consider the first term $\|D^s f\|_{L^p} \|g\|_{L^q}$ from the right hand side. $\|g\|_{L^q}$ is independent of k , while

$$\begin{aligned} [D^s f](x) &= \int_{\mathbf{R}^n} |\xi|^s \Psi(2^{-k} \xi) \Phi(\xi - 2^k e_1) e^{2\pi i \langle \xi, x \rangle} d\xi \\ &= 2^{ks} \int_{\mathbf{R}^n} \Psi_s(2^{-k} \xi) \Phi(\xi - 2^k e_1) e^{2\pi i \langle \xi, x \rangle} d\xi \end{aligned}$$

where $\Psi_s(\cdot) := |\cdot|^s \Psi(\cdot)$. Thus we have

$$\|D^s f\|_{L^p} = 2^{ks} \left\| 2^{kn} [\widehat{\Psi}_s(2^k \cdot)] * [e^{2\pi i \langle 2^k e_1, \cdot \rangle} \widehat{\Phi}] \right\|_{L^p} \leq 2^{ks} \left\| \widehat{\Psi}_s \right\|_{L^1} \left\| \widehat{\Phi} \right\|_{L^p}.$$

Note that since Ψ is supported on an annulus, $\Psi_s \in \mathcal{S}(\mathbf{R}^n)$ so that $\widehat{\Psi}_s \in \mathcal{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$. Thus this term converges to zero as $k \rightarrow \infty$. The second term on the right hand side of (2) is estimated similarly, which leads to a contradiction. \square

Proof of Theorem 1 in the homogeneous case. Define $\Phi \in \mathcal{S}(\mathbf{R})$ so that $\Phi \equiv 1$ on $[-1, 1]$ and is supported in $[-2, 2]$. Also let $\Psi(\xi) := \Phi(\xi) - \Phi(2\xi)$ and note that Ψ is supported on an annulus $\xi : 1/2 < |\xi| < 2$ and $\sum_{k \in \mathbf{Z}} \Psi(2^{-k} \xi) = 1$ for $\forall \xi \neq 0$.

Given $f, g \in \mathcal{S}(\mathbf{R}^n)$, we decompose $D^s[f g]$ as follows:

$$\begin{aligned} D^s[f g](x) &= \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \left(\sum_{j \in \mathbf{Z}} \Psi(2^{-j} \xi) \widehat{f}(\xi) \right) \left(\sum_{k \in \mathbf{Z}} \Psi(2^{-k} \eta) \widehat{g}(\eta) \right) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \sum_{j \in \mathbf{Z}} \sum_{k: k < j-1} \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \Psi(2^{-j} \xi) \widehat{f}(\xi) \Psi(2^{-k} \eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \end{aligned}$$

$$\begin{aligned}
& + \sum_{k \in \mathbf{Z}} \sum_{j: j < k-1} \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \Psi(2^{-j}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
& + \sum_{k \in \mathbf{Z}} \sum_{j: |j-k| \leq 1} \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \Psi(2^{-j}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
& =: \Pi_1[f, g](x) + \Pi_2[f, g](x) + \Pi_3[f, g](x).
\end{aligned}$$

The arguments for Π_1 and Π_2 are identical under the apparent symmetry, so it suffices to consider Π_1 and Π_3 . For Π_1 , we can write

$$\Pi_1[f, g](x) = \int_{\mathbf{R}^{2n}} \left\{ \sum_{j \in \mathbf{Z}} \Psi(2^{-j}\xi) \Phi(2^{-j+2}\eta) \frac{|\xi + \eta|^s}{|\xi|^s} \right\} \widehat{D^s f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta.$$

Since the expression in the bracket above is a bilinear Coifman-Meyer multiplier, the $\Pi_1[f, g]$ satisfies the inequality (2).

For $\Pi_3[f, g]$, note that the summation in j is finite, thus it suffices to show estimate (2) for the term

$$(16) \quad \left\| \sum_{k \in \mathbf{Z}} \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \right\|_{L^r(\mathbf{R}^n)}.$$

When $s \in 2\mathbf{N}$, (16) can be written as

$$\left\| \int_{\mathbf{R}^{2n}} \left\{ \sum_{k \in \mathbf{Z}} \frac{|\xi + \eta|^s}{|\eta|^s} \Psi(2^{-k}\xi) \Psi(2^{-k}\eta) \right\} \widehat{f}(\xi) \widehat{D^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \right\|_{L^r(\mathbf{R}^n)}.$$

The expression in the bracket above belongs to Coifman-Meyer class, so the theorem follows directly in this case. When $s \notin 2\mathbf{N}$, the symbol at hand is rougher than what is permitted from the current multilinear Fourier multiplier theorem. At the remark at the end of this section we explain why this symbol cannot be treated by known multiplier theorems.

We proceed with the estimate for Π_3 , which requires a more careful analysis. We have the following cases.

Case 1: $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$ or $\frac{1}{2} \leq r < 1$, $1 \leq p, q < \infty$.

These represent two separate cases: the former has the strong L^r norm on the left hand side of (2), and the latter has the weak L^r norm instead. However, in view of Theorem A and Corollary 1, the strategy for the proof will be identical. Thus we will only prove the estimate with a strong L^r norm on the left hand side. Notice that when $|\xi|, |\eta| \leq 2 \cdot 2^k$, then $|\xi + \eta| \leq 2^{k+2}$ and thus $\Phi(2^{-k-2}(\xi + \eta)) = 1$. In view of this **one term of Π_3 looks like**

$$\begin{aligned}
& \Pi_3[f, g](x) \\
& = \iint_{\mathbf{R}^{2n}} \sum_{k \in \mathbf{Z}} |\xi + \eta|^s \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
& = \iint_{\mathbf{R}^{2n}} \sum_{k \in \mathbf{Z}} |\xi + \eta|^s \Phi(2^{-k-2}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta
\end{aligned}$$

$$\begin{aligned}
&= 2^{2s} \sum_{k \in \mathbf{Z}} \iint_{\mathbf{R}^{2n}} \Phi_s(2^{-k-2}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) \widetilde{\Psi}(2^{-k}\eta) \widehat{D^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
&= 2^{2s} \sum_{k \in \mathbf{Z}} 2^{2nk} \iint_{\mathbf{R}^{2n}} \Phi_s(2^{-2}(\xi + \eta)) \Psi(\xi) \widehat{f}(2^k \xi) \widetilde{\Psi}(\eta) \widehat{D^s g}(2^k \eta) e^{2\pi i 2^k \langle \xi + \eta, x \rangle} d\xi d\eta,
\end{aligned}$$

where $\widetilde{\Psi}(\cdot) := |\cdot|^{-s} \Psi(\cdot)$, $\Phi_s(\cdot) := |\cdot|^s \Phi(\cdot)$. Now the function $\Phi_s(2^{-2}\cdot)$ is supported in $[-8, 8]^n$ and can be expressed in terms of its Fourier series multiplied by the characteristic function of the set $[-8, 8]^n$, denoted $\chi_{[-8, 8]^n}$.

$$\Phi_s(2^{-2}(\xi + \eta)) = \sum_{m \in \mathbf{Z}^n} c_m^s e^{\frac{2\pi i}{16} \langle \xi + \eta, m \rangle} \chi_{[-8, 8]^n}(\xi + \eta),$$

where $c_m^s := \frac{1}{16^n} \int_{[-8, 8]^n} |y|^s \Phi(2^{-2}y) e^{-\frac{2\pi i}{16} \langle y, m \rangle} dy$. Due to the support of Ψ and $\widetilde{\Psi}$, we also have

$$\chi_{[-8, 8]^n}(\xi + \eta) \Psi(\xi) \widetilde{\Psi}(\eta) = \Psi(\xi) \widetilde{\Psi}(\eta),$$

so that the characteristic function may be omitted from the integrand. Using this identity, we write $\Pi_3[f, g](x)$ as

$$\begin{aligned}
&= 2^{2s} \sum_{k \in \mathbf{Z}} 2^{2nk} \iint_{\mathbf{R}^{2n}} \sum_{m \in \mathbf{Z}^n} c_m^s e^{\frac{2\pi i}{16} \langle \xi + \eta, m \rangle} \Psi(\xi) \widehat{f}(2^k \xi) \widetilde{\Psi}(\eta) \widehat{D^s g}(2^k \eta) e^{2\pi i 2^k \langle \xi + \eta, x \rangle} d\xi d\eta \\
&= 2^{2s} \sum_{m \in \mathbf{Z}^n} c_m^s \sum_{k \in \mathbf{Z}} [\Delta_k^m f](x) [\widetilde{\Delta}_k^m D^s g](x),
\end{aligned}$$

where Δ_k^m is the Littlewood-Paley operator given by multiplication on the Fourier transform side by $e^{2\pi i \langle 2^{-k}\cdot, \frac{m}{16} \rangle} \Psi(2^{-k}\cdot)$, while $\widetilde{\Delta}_k^m$ is the Littlewood-Paley operator given by multiplication on the Fourier side by $e^{2\pi i \langle 2^{-k}\cdot, \frac{m}{16} \rangle} \widetilde{\Psi}(2^{-k}\cdot)$. Both Littlewood-Paley operators have the form:

$$\int_{\mathbf{R}^n} 2^{nk} \Theta(2^k(x - y) + \frac{1}{16}m) f(y) dy$$

for some Schwartz function Θ whose Fourier transform is supported in some annulus centered at zero.

Let $r_* := \min(r, 1)$. Taking the L^r norm of the right hand side above, we obtain

$$\begin{aligned}
&\|D^s[f, g]\|_{L^r}^{r_*} \\
&\leq \sum_{m \in \mathbf{Z}^n} |c_m^s|^{r_*} \left\| \sum_{k \in \mathbf{Z}} [\Delta_k^m f](x) [\widetilde{\Delta}_k^m D^s g](x) \right\|_{L^r(\mathbf{R}^n)}^{r_*} \\
&\leq \sum_{m \in \mathbf{Z}^n} |c_m^s|^{r_*} \left\| \sqrt{\sum_{k \in \mathbf{Z}} |\Delta_k^m f|^2} \right\|_{L^p(\mathbf{R}^n)}^{r_*} \left\| \sqrt{\sum_{k \in \mathbf{Z}} |\widetilde{\Delta}_k^m D^s g|^2} \right\|_{L^q(\mathbf{R}^n)}^{r_*}
\end{aligned}$$

whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. By Corollary 1, the preceding expression is bounded by a constant multiple of

$$\sum_{m \in \mathbf{Z}^n} |c_m^s|^{r_*} [\ln(2 + |m|)]^{2r_*} \|f\|_{L^p}^{r_*} \|D^s g\|_{L^q}^{r_*}$$

if $1 < p, q < \infty$ and this term yields a constant, provided we can show that the series above converges.

Now, applying Lemma 1,

$$c_m^s = \int_{[-8,8]^n} |\xi|^s \Phi(2^{-2}\xi) e^{-\frac{2\pi i}{16}\langle m, \xi \rangle} d\xi = c [D^s \widehat{\Phi(2^{-2}\cdot)}] \left(\frac{m}{16}\right) = O((1 + |m|)^{-n-s})$$

as $|m| \rightarrow \infty$ and c_m^s is uniformly **decaying** for all $m \in \mathbf{Z}$. Thus, since $r_*(n+s) > n$, the series $\sum_{m \in \mathbf{Z}^n} |c_m^s|^{r_*} [\ln(1 + |m|)]^{2r_*}$ converges. This concludes Case 1.

Case 2: $1 < r < \infty$, $(p, q) \in \{(r, \infty), (\infty, r)\}$

Here we provide a proof, which is an adaptation of the proof given in [1, Section 6.2]. This method will extend more readily to the multi-parameter case, which is presented in Section 5. Write

$$\|\Pi_3[f, g]\|_{L^r(\mathbf{R}^n)} \leq C(r, n) \left\| \left(\sum_{j \in \mathbf{Z}} |\Delta_j \Pi_3[f, g]|^2 \right)^{\frac{1}{2}} \right\|_{L^r(\mathbf{R}^n)}.$$

The summand in j above can be estimated **for one part of Π_3** as follows

$$\begin{aligned} & \Delta_j \Pi_3[f, g](x) \\ &= \int_{\mathbf{R}^{2n}} |\xi + \eta|^s \Psi(2^{-j}(\xi + \eta)) \sum_{k \geq j-2} \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \int_{\mathbf{R}^{2n}} 2^{js} \widetilde{\Psi}_s(2^{-j}(\xi + \eta)) \widehat{f}(\xi) \sum_{k \geq j-2} 2^{-ks} \Psi(2^{-k}\xi) \widetilde{\Psi}_{-s}(2^{-k}\eta) \widehat{D^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= 2^{js} \sum_{k \geq j-2} 2^{-ks} \widetilde{\Delta}_j^s \left[[\Delta_k f] [\widetilde{\Delta}_k^{-s} D^s g] \right] (x) \\ &\leq 2^{js} \left(\sum_{k \geq j-2} 2^{-2ks} \right)^{\frac{1}{2}} \left(\sum_{k \geq j-2} \left| \widetilde{\Delta}_j^s \left[[\Delta_k f] [\widetilde{\Delta}_k^{-s} D^s g] \right] (x) \right|^2 \right)^{\frac{1}{2}} \\ &\leq C(s) \left(\sum_{k \geq j-2} \left| \widetilde{\Delta}_j^s \left[[\Delta_k f] [\widetilde{\Delta}_k^{-s} D^s g] \right] (x) \right|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where $\widetilde{\Psi}_s(\cdot) := |\cdot|^s \Psi(\cdot)$ and $\mathcal{F}[\widetilde{\Delta}_k^s f](\cdot) := \widetilde{\Psi}_s(2^{-k}\cdot) \widehat{f}(\cdot)$. Thus we have

$$\|\Pi_3[f, g]\|_{L^r} \leq C(r, n, s) \left\| \left(\sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} \left| \widetilde{\Delta}_j^s [\Delta_k f \widetilde{\Delta}_k^{-s} D^s g] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r}.$$

We apply [10, Proposition 4.6.4] to extend $\{\widetilde{\Delta}_k^s\}_{k \in \mathbf{Z}}$ from $L^r \rightarrow L^r \ell^2$ to $L^r \ell^2 \rightarrow L^r \ell^2 \ell^2$ for $1 < r < \infty$. This gives

$$\|\Pi_3[f, g]\|_{L^r} \leq C(r, n, s) \left\| \left(\sum_{k \in \mathbf{Z}} \left| \Delta_k f \widetilde{\Delta}_k D^s g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r}$$

$$\begin{aligned}
&\leq C(r, n, s) \left\| \sup_{k \in \mathbf{Z}} \widetilde{\Delta}_k D^s g \right\|_{L^\infty} \left\| \left(\sum_{k \in \mathbf{Z}} |\Delta_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \\
&\leq C(r, n, s) \sup_{k \in \mathbf{Z}} \left\| \widetilde{\Delta}_k D^s g \right\|_{L^\infty} \|f\|_{L^r} \\
&\leq C(r, n, s) \left\| \widehat{\Psi}_{-s} \right\|_{L^1} \|D^s g\|_{L^\infty} \|f\|_{L^r}.
\end{aligned}$$

This proves the case when $(p, q) = (r, \infty)$ and the case $(p, q) = (\infty, r)$ follows by symmetry. \square

We emphasize that the bound for Π_3 does not follow from presently-known multiplier estimates. In the following remark, we consider the symbol of Π_3 under the smoothness criteria given in [2, 6, 12, 20]. Define

$$\sigma_s(\xi, \eta) := \sum_{k \in \mathbf{Z}} \frac{|\xi + \eta|^s}{|\xi|^s} \Psi(2^{-k}\xi) \Psi(2^{-k}\eta).$$

Then $\Pi_3[f, g] = T_{\sigma_s}[f, D^s g]$, where T_{σ_s} is the pseudo-differential operator with symbol σ_s . We observe the following.

Remark 1. For $s \in \mathbf{R}_+ \setminus 2\mathbf{N}$,

- (1) σ_s does not belong to the Coifman-Meyer class (Theorem A) for $s < 2n$.
- (2) $\sigma_s \in \dot{B}S_{1,0;-\pi/4}^0$ defined below, which forms a degenerate class of pseudo-differential operators, [2].
- (3) σ_s fail the smoothness conditions in [12] if $s < n/2$.

Sketch of proof of Remark 1. Firstly, it is easily seen that $\partial^\alpha \sigma_s$ develops a singularity on the hyperplane $\xi + \eta = 0$ whenever $|\alpha| > s$, so the bound $|\partial_{\xi_1}^{2n+1} \sigma_s|(\xi, \eta) \leq C(|\xi| + |\eta|)^{-2n-1}$ from Theorem A is not satisfied when $s < 2n$ for any $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$ with $\xi + \eta = 0$.

Secondly, we recall the class of bilinear symbols denoted $BS_{0,1;\theta}^0$ for $\theta \in (-\pi/2, \pi/2]$ given in [2]: $\sigma(\xi, \eta) \in BS_{0,1;\theta}^0$ if

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha,\beta} (1 + |\eta - \xi \tan \theta|)^{-|\alpha| - |\beta|};$$

and $\sigma \in \dot{B}S_{1,0;\theta}^0$ if the right hand side above can be replaced by $C_{\alpha,\beta} |\eta - \xi \tan \theta|^{-|\alpha| - |\beta|}$.

In [2], Bényi, Nahmod, Torres remarked that the classes of symbols $BS_{1,0;\theta}^0$ and $\dot{B}S_{1,0;\theta}^0$ given by the angle $\theta = -\pi/4$ (along with $\theta = 0$ and $\theta = \pi/2$) are degenerate in the sense that the boundedness results for singular multipliers do not apply anymore, even for $r > 1$. As we noted above, σ_s develops a singularity along $\xi + \eta = 0$, which is permitted only in the class $\dot{B}S_{1,0;-\pi/4}^0$. So we can easily note that $\sigma_s \in \dot{B}S_{0,1;\theta}^0$ if and only if $\theta = -\pi/4$. This portrays that σ_s does not belong to a known bounded class of singular multiplier symbols.

Lastly, we show that σ_s fails to satisfy the condition given in [20] and also in [12] when $s < n$. Let $\Psi \in \mathcal{S}(\mathbf{R}^{2n})$ be defined such that Ψ is supported on an annulus

$1/2 \leq |(\xi, \eta)| \leq 2$ and $\sum_{j \in \mathbf{Z}} \Psi(2^{-j}\xi, 2^{-j}\eta) = 1$ except at the origin. The condition in [20] and [12] requires that for some $\gamma > n$,

$$\sup_{j \in \mathbf{Z}} \|\Psi(\cdot)\sigma_s(2^j \cdot)\|_{L^2_\gamma(\mathbf{R}^{2n})} < \infty.$$

We will show that right hand side of above is infinite when $s \in \mathbf{R}_+ \setminus 2\mathbf{Z}_+$, $s < n/2$ and $\gamma = n$.

Note that $\sigma_s(2^j\xi, 2^j\eta) = \sigma_s(\xi, \eta)$ for any $j \in \mathbf{Z}$ and also that multiplying Ψ to σ_s reduces the summation in k to a finite sum $\sum_{|k| \leq 1}$. Thus,

$$\Psi(\xi, \eta)\sigma_s(2^j\xi, 2^j\eta) = |\xi + \eta|^s \tilde{\Psi}(\xi, \eta)$$

where

$$\widehat{\Psi}(\xi, \eta) := \Psi(\xi, \eta) \sum_{|k| \leq 1} |\xi|^{-s} \Psi(2^{-k}\xi) \Psi(2^{-k}\eta).$$

Note that for any $s \in \mathbf{R} \setminus \{0\}$,

$$\Delta_\xi |\xi + \eta|^s = s(s + n - 2)|\xi + \eta|^{s-2} \quad |\nabla_\xi |\xi + \eta|^s| = s|\xi + \eta|^{s-1}.$$

For the derivatives on $|\xi + \eta|^s \tilde{\Psi}(\xi, \eta)$, the singularity along the hyperplane $\{\xi + \eta = 0\}$ is dominated by $\partial^\alpha |\xi + \eta|^s$ when $|\alpha| > s$. For our purpose, we can assume (near this hyperplane) that all derivatives fall on the rough term $|\xi + \eta|^s$.

First, let $n = 2k$ for some $k \in \mathbf{N}$. To see that $|\xi + \eta|^s \tilde{\Psi}(\xi, \eta) \notin L^2_n(\mathbf{R}^{2n})$, there is some $C(s, n) \neq 0$ such that

$$[\Delta_\xi^k |\xi + \eta|^s] \tilde{\Psi}(\xi, \eta) = C(s, n) |\xi + \eta|^{s-n} \tilde{\Psi}(\xi, \eta).$$

Also note that $\tilde{\Psi}$ is bounded below by some constant within the set

$$\left\{ (\xi, \eta) \in \mathbf{R}^{2n} : |\xi + \eta| \leq \frac{1}{10}, \frac{3}{4} \leq |\eta| \leq \frac{3}{2} \right\}$$

Thus,

$$\begin{aligned} \|\Phi\sigma_s\|_{L^2_n} &\geq C(n) \int_{\mathbf{R}^{2n}} \left| \Delta_\xi^k [|\xi + \eta|^s] \tilde{\Psi}(\xi, \eta) \right|^2 d\xi d\eta \\ &\geq C(s, n) \int_{\eta: \frac{3}{4} \leq |\eta| \leq \frac{3}{2}} \int_{\xi: |\xi + \eta| \leq \frac{1}{10}} |\xi + \eta|^{2s-2n} d\xi d\eta \\ &= C(s, n) \int_{|\xi| \leq \frac{1}{10}} |\xi|^{2s-2n} d\xi \end{aligned}$$

which is infinite when $2s - 2n < -n$ (i.e. $s < n/2$).

If $n = 2k + 1$ for some $k \in \mathbf{N}$, then we can make the same argument as above after replacing Δ_ξ^k by $|\nabla_\xi [\Delta_\xi^k |\xi + \eta|^s]|$. \square

4. INHOMOGENEOUS KATO-PONCE INEQUALITY

In this section, we discuss the original Kato-Ponce inequality (3). Our approach is largely based on the idea developed in the previous section for the homogeneous symbol and does not depend on the smoothness of the symbol $(1+|\cdot|)^{s/2}$. It is indeed surprising that not only the positive results, but also the negative results parallel the ones given for the homogeneous symbol. This phenomenon is sharply observed in the following lemma.

Lemma 2. *Let $f \in \mathcal{S}(\mathbf{R}^n)$ and $s > 0$. Then for all $\delta \in (0, 1]$, there exists a constant $C(n, s, f)$ independent of δ such that*

$$(17) \quad |(\delta^2 - \Delta)^{s/2} f|(x) \leq C(n, s, f)(1 + |x|)^{-n-s}$$

Remark: In [7], the authors use scaling of (3) and dominated convergence to deduce (2) from (3). This lemma gives a justification for this argument as well. In fact, these two inequalities are intricately related due to the heuristic relationship

$$J^s := (1 - \Delta)^{s/2} \approx 1 + (-\Delta)^{s/2} =: 1 + D^s.$$

This approximation is rigorous when working in L^p for $1 \leq p < \infty$, so that (2) may imply (3) as well in this case.

Proof. First, note that $[(\delta^2 - \Delta)^{s/2} f]$ is uniformly bounded in $\delta \in (0, 1]$ since

$$\left| \int_{\mathbf{R}^n} (\delta^2 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi \right| \leq \int_{\mathbf{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} |\widehat{f}|(\xi) d\xi < \infty.$$

It remains to show that for $|x| \geq 1$, $|(\delta^2 - \Delta)^{s/2} f|(x) \leq C(n, s, f)|x|^{-n-s}$.

For $z \in \mathbf{C}$ and $\delta \in (0, 1]$, define the distribution v_z^δ by the action

$$\langle v_z^\delta, f \rangle = \int_{\mathbf{R}^n} (\delta^2 + |\xi|^2)^{\frac{z}{2}} f(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbf{R}^n)$. Note that the map $z \mapsto \langle v_z^\delta, f \rangle$ defines an entire function. If $z \in \mathbf{R}$ and $z < 0$ and $\delta = 1$, $\widehat{v_z^\delta}$ is known as the Bessel potential, denoted G_{-z} , given in [10, Chapter 6]. We now extend the distribution $\widehat{v_z^\delta}$ to $z \in \mathbf{C}$.

Begin with the Gamma function identity: for $A > 0$ and $z : \operatorname{Re} z < 0$

$$A^z = \frac{1}{\Gamma(-z)} \int_0^\infty e^{tA} t^{-z-1} dt.$$

Consider the map $z \mapsto \langle v_z^\delta, \widehat{f} \rangle$ when $z : \operatorname{Re} z < 0$. Using the identity above, we have

$$\begin{aligned} \langle v_z^\delta, \widehat{f} \rangle &:= \int_{\mathbf{R}^n} \frac{1}{\Gamma(-\frac{z}{2})} \int_0^\infty e^{-\delta^2 t} e^{-t|\xi|^2} t^{-\frac{z}{2}-1} dt \widehat{f}(\xi) d\xi \\ &= \frac{1}{\Gamma(-\frac{z}{2})} \int_{\mathbf{R}^n} \int_0^\infty e^{-\frac{|y|^2}{t}} f(y) e^{-\delta^2 t} t^{-\frac{z+n}{2}} \frac{dt}{t} dy. \end{aligned}$$

Denote $K_z^\delta(y) := \int_0^\infty e^{-\frac{|y|^2}{t}} e^{-\delta^2 t} t^{-\frac{z+n}{2}} \frac{dt}{t}$ to be the kernel above. First we observe,

$$|K_z^\delta|(y) = |y|^{-\operatorname{Re} z - n} \left| \int_0^\infty e^{-\frac{1}{t}} e^{-\delta^2 |y|^2 t} t^{-\frac{z+n}{2}} \frac{dt}{t} \right| \leq C(z, n) |y|^{-\operatorname{Re} z - n}$$

for all $y \in \mathbf{R}^n \setminus \{0\}$ when $\operatorname{Re} z > -n$. Recall from [10, Proposition 6.1.5] that $|K_z^\delta|(y) \sim \ln |y|^{-1}$ when $\operatorname{Re} z = -n$; and $|K_z^\delta|(y) \sim 1$ when $\operatorname{Re} z < -n$.

Given $\delta > 0$, the kernel also satisfies a better asymptotic estimate for sufficiently large $|y|$. For $|y| > \delta^{-1}$ and $\delta \leq 1$, note that $\delta^2 t + \frac{|y|^2}{t} \geq \max(\delta^2 t + \frac{1}{\delta^2 t}, 2\delta|y|)$. Thus

$$|K_z^\delta|(y) \leq e^{-\delta|y|} \int_0^\infty e^{-\frac{1}{2}\delta^2 t + \frac{1}{2\delta^2 t}} t^{-\frac{\operatorname{Re} z + n}{2}} \frac{dt}{t} = \delta^{\operatorname{Re} z + n} e^{-\delta|y|} \int_0^\infty e^{-\frac{1}{2}(t + \frac{1}{t})} t^{-\frac{\operatorname{Re} z + n}{2}} \frac{dt}{t}.$$

Since the integral above converges for any $z \in \mathbf{C}$, we have that for all $|y| > \delta^{-1}$, $|K_z^\delta|(y) \leq C(z, n) \delta^{\operatorname{Re} z + n} e^{-\delta|y|}$. We remark that $\sup_{\delta \in (0, 1]} \delta^{\operatorname{Re} z + n} e^{-\delta|y|} = c |y|^{-n - \operatorname{Re} z}$.

Note that $K_z(y)$ is not locally integrable when $\operatorname{Re} z \geq 0$ so that it is not well-defined as a tempered distribution. However, since $z \mapsto \langle v_z^\delta, \widehat{f} \rangle$ is an entire function, it suffices to find a holomorphic extension of $\langle \widehat{v}_z^\delta, f \rangle$ which is defined as $\langle K_z^\delta, f \rangle$ for $\operatorname{Re} z < 0$. We continue

$$\begin{aligned} (18) \quad \langle \widehat{v}_z^\delta, f \rangle &= \frac{1}{\Gamma(-\frac{z}{2})} \int_{\mathbf{R}^n} K_z^\delta(y) \left[f(y) - \sum_{|\alpha| < N} \frac{[\partial^\alpha f](0)}{\alpha!} y^\alpha \right] dy \\ &\quad + \sum_{|\alpha| < N} \frac{[\partial^\alpha f](0)}{\alpha!} \frac{1}{\Gamma(-\frac{z}{2})} \int_{\mathbf{R}^n} y^\alpha \int_0^\infty e^{-\frac{|y|^2}{t}} e^{-\delta^2 t} t^{-\frac{z+n}{2}} \frac{dt}{t} dy \\ &=: I_1^\delta(z) + I_2^\delta(z). \end{aligned}$$

Consider first I_2^δ .

$$\begin{aligned} (19) \quad I_2^\delta(z) &= \sum_{|\alpha| < N} \frac{[\partial^\alpha f](0)}{\alpha!} \frac{1}{\Gamma(-\frac{z}{2})} \int_{\mathbf{R}^n} y^\alpha \int_0^\infty e^{-\frac{|y|^2}{t}} e^{-\delta^2 t} t^{-\frac{z+n}{2}} \frac{dt}{t} dy \\ &= \sum_{|\alpha| < N} \frac{[\partial^\alpha f](0)}{\alpha!} \frac{1}{\Gamma(-\frac{z}{2})} \int_0^\infty e^{-\delta^2 t} t^{\frac{|\alpha| - z}{2}} \frac{dt}{t} \int_{\mathbf{R}^n} y^\alpha e^{-|y|^2} dy \\ &= \sum_{|\alpha| < N} \frac{\delta^{z - |\alpha|}}{\alpha!} [\partial^\alpha f](0) \frac{1}{\Gamma(-\frac{z}{2})} \int_0^\infty e^{-t} t^{\frac{|\alpha| - z}{2}} \frac{dt}{t} \int_{\mathbf{R}^n} y^\alpha e^{-|y|^2} dy \\ &= \sum_{|\alpha| < N} \frac{\delta^{z - |\alpha|}}{\alpha!} [\partial^\alpha f](0) \frac{\Gamma\left(\frac{|\alpha| - z}{2}\right)}{\Gamma(-\frac{z}{2})} \int_0^\infty r^{n + |\alpha| - 1} e^{-r^2} dr \int_{S^{n-1}} \theta^\alpha d\theta. \end{aligned}$$

Note that the integral $\int_{S^{n-1}} \theta^\alpha d\theta$ vanishes unless α_j is even for all $j = 1, 2, \dots, n$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. In this case, $|\alpha|$ is even. Therefore, the poles of the function $\Gamma((|\alpha| - z)/2)$ cancel with the poles of $\Gamma(-z/2)$, and thus $I_2^\delta(z)$ is entire.

We should remember that when z is a positive even integer, the poles of $\Gamma(-\cdot/2)$ make the term $I_1^\delta(z)$ vanish identically. Thus, as expected in this case, \widehat{v}_z^δ yields a local differential operators composed of only even-order derivatives.

Next we turn our attention to $I_1^\delta(z)$. The expression inside the square bracket on the right hand side of (18) is locally $O(|y|^N)$. Since $K_z^\delta(y) = O(|y|^{-n-\operatorname{Re}z})$, the integral converges locally (say $|y| \leq \delta^{-1}$) if $\operatorname{Re}z < N$. For $|y| > \delta^{-1}$, the kernel $K_z^\delta(y)$ decays exponentially, while the expression inside the square bracket grows at most like $O(|y|^{N-1})$. Thus, the integral converges.

Therefore, we note that (18) and (19) extends the function $z \mapsto \langle K_z, f \rangle$ holomorphically on the half plane $z : \operatorname{Re}z < N$. Thus, this defines the tempered distribution \widehat{v}_z^δ .

Now consider $f_\delta^s := (\delta^2 - \Delta)^{s/2} f$ for $s > 0$ and $\delta > 0$. We can write $f_\delta^s(x) = \langle v_s^\delta, \widehat{f}e^{i(\cdot, x)} \rangle = \langle \widehat{v}_s^\delta, f(\cdot + x) \rangle$. Let $s \in [N-1, N)$ for some $N \in \mathbf{N}$. Using the formula (18) and (19), $\langle \widehat{v}_s^\delta, f(\cdot + x) \rangle$ can be expressed as

$$C(s) \int_{\mathbf{R}^n} K_s^\delta(y) \left[f(x+y) - \sum_{|\alpha| < N} \frac{[\partial^\alpha f](x)}{\alpha!} y^\alpha \right] dy + \sum_{|\alpha| < N} C(\alpha, n) \delta^{s-|\alpha|} [\partial^\alpha f](x),$$

where the first constant $C(s) = 0$ when s is a positive even integer.

The second term above is a Schwartz function, and decays uniformly in $\delta \in (0, 1]$ since $s - |\alpha| \geq 0$ when $|\alpha| \leq N-1$.

For the first term, we split the integral into two parts $\int_{|y| < 1} \cdot dy + \int_{|y| \geq 1} \cdot dy =: J_1(x) + J_2(x)$. We have that

$$\begin{aligned} J_1(x) &= \int_{|y| < 1} K_s^\delta(y) \left[f(x+y) - \sum_{|\alpha| < N} [\partial^\alpha f](x) y^\alpha \right] dy \\ &\leq \sup_{|\beta|=N} \sup_{|y'| < 1} |\partial^\beta f|(x+y') \int_{|y| < 1} |y|^{-n-s+N} dy. \end{aligned}$$

Since $-n-s+N > -n$, the last integral above is convergent. Also, we note that the expression $\sup_{|\beta|=N} \sup_{|y'| < 1} |\partial^\beta f|(x+y')$ decays like a Schwartz function.

The estimate for J_2 is more delicate. We need to consider separately the case $s = N-1$ and $s \in (N-1, N)$. First, consider when $s \in (N-1, N)$. In this case,

$$\begin{aligned} J_2(x) &= \int_{|y| \geq 1} K_s^\delta(y) \left[f(x+y) - \sum_{|\alpha| < N} [\partial^\alpha f](x) y^\alpha \right] dy \\ &= \int_{|y| \geq 1} |y|^{-n-s} |f|(x+y) dy + \sum_{|\alpha| \leq N-1} |\partial^\alpha f|(x) \int_{|y| \geq 1} |y|^{-n-s+|\alpha|} dy. \end{aligned}$$

For the first term

$$\int_{|y| \geq 1} |y|^{-n-s} |f|(x+y) dy \leq |x|^{-M} \int_{2|y| \leq |x|} |x+y|^M |f|(x+y) dy$$

$$+ C|x|^{-n-s} \int_{2|y|>|x|} |f|(x+y) dy.$$

for any $M \in \mathbf{N}$. Thus this decays like $|x|^{-n-s}$.

For the second term, the integral $\int_{|y|\geq 1} |y|^{-n-s+|\alpha|} dy$ converges since $-n-s+|\alpha| < -n$ when $|\alpha| \leq N+1$ and $s \in (N-1, N)$, so that the second term decays like a Schwartz function.

Now consider the special case when $s = N-1$. Note that s has to be an odd integer, since otherwise, the terms J_1, J_2, J_3 would not even appear due to the vanishing constant $C(s)$ mentioned above. Since $K_z^\delta(y)$ is a radial function (and exponentially decaying), the integral

$$\int_{|y|\geq 1} K_z^\delta(y) y^\alpha dy = \int_1^\infty K_z^\delta(r) r^{|\alpha|+n-1} dr \int_{S^{n-1}} \theta^\alpha d\theta = 0$$

since $|\alpha|$ is odd. This concludes the proof of Lemma 2. \square

Proof of Theorem 2 for the inhomogeneous case. Fix an index $\frac{1}{2} < r < 1$ and indices $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r} = \frac{1}{p_2} + \frac{1}{q_2}$; also fix $0 < s \leq \frac{n}{r} - n$.

Assume that (3) holds and we will reach a contradiction. Scaling $x \mapsto \lambda x$ for some $\lambda > 0$, this inequality is equivalent to

$$(20) \quad \begin{aligned} & \|(\lambda^{-2} - \Delta)^{s/2}[fg]\|_{L^r} \leq \\ & C \left(\|(\lambda^{-2} - \Delta)^{s/2}f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|(\lambda^{-2} - \Delta)^{s/2}g\|_{L^{q_2}} \right). \end{aligned}$$

By Lemma 2, we know that the functions $[(\lambda^{-2} - \Delta)^{s/2}f](x)$ and $[(\lambda^{-2} - \Delta)^{s/2}g](x)$ are pointwise dominated by a constant multiple of $(1+|x|)^{-n-s}$ uniformly for $\lambda > 1$. Then the right hand side of (20) is bounded uniformly in $\lambda > 1$.

On the other hand, $(\lambda^{-2} - \Delta)^{s/2}[fg] \rightarrow D^s[fg]$ pointwise everywhere by Lebesgue dominated convergence. By Fatou's lemma, this implies that

$$\int_{\mathbf{R}^n} |D^s[fg]|^r dx \leq \liminf_{\lambda \rightarrow \infty} \int_{\mathbf{R}^n} |(\lambda^{-2} - \Delta)^{s/2}[fg]|^r dx.$$

Since $D^s[fg] \notin L^r(\mathbf{R}^n)$ if $0 < s \leq n/r - n$, the left hand side of (20) is infinite, which leads to a contradiction.

When $\frac{n}{r} - n < s < 0$, consider the counter-example given in Section 3. The left hand side of (3) is independent of k , while the $\|J^s f\|_{L^p}$ term on the right side can be written as

$$\begin{aligned} [J^s f](x) &= \int_{\mathbf{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} \Phi(\xi - 2^k e_1) e^{2\pi i \xi \cdot x} d\xi \\ &= \int_{\mathbf{R}^n} 2^{ks} \Psi_s^k(\xi) \Phi(\xi - 2^k e_1) e^{2\pi i \xi \cdot x} d\xi \end{aligned}$$

where $\Psi_s^k(\cdot) := (2^{-2k} + |\xi|^2)^{\frac{s}{2}} \Psi(\cdot)$. Then

$$\|J^s f\|_{L^p} \leq 2^{ks} \|\widehat{\Psi_s^k}\|_{L^1} \|\widehat{\Phi}\|_{L^p}.$$

The fact that $\|\widehat{\Psi_s^k}\|_{L^1}$ is uniformly bounded is shown below in Lemma 3 and the remark following. Taking $k \rightarrow \infty$, we arrive at a contradiction. \square

Proof of Theorem 1 for the inhomogeneous case. We resume the notations Φ, Ψ introduced in Section 3. Via similar computations, we split the estimate above into Π_1, Π_2, Π_3 . More precisely,

$$\begin{aligned} J^s[f g](x) &= \sum_{j \in \mathbf{Z}} \sum_{k: k < j-1} \int_{\mathbf{R}^{2n}} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Psi(2^{-j}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &\quad + \sum_{k \in \mathbf{Z}} \sum_{j: j < k-1} \int_{\mathbf{R}^{2n}} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Psi(2^{-j}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &\quad + \sum_{k \in \mathbf{Z}} \sum_{j: |j-k| \leq 1} \int_{\mathbf{R}^{2n}} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Psi(2^{-j}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &=: \Pi_1[f, g](x) + \Pi_2[f, g](x) + \Pi_3[f, g](x). \end{aligned}$$

As described in Section 3, estimates for Π_1 and Π_2 follow from Theorem A. More specifically,

$$\Pi_1[f, g](x) = \int_{\mathbf{R}^{2n}} \left\{ \sum_{j \in \mathbf{Z}} \Psi(2^{-j}\xi) \Phi(2^{-j+2}\eta) \frac{(1 + |\xi + \eta|^2)^{\frac{s}{2}}}{(1 + |\xi|^2)^{\frac{s}{2}}} \right\} \widehat{J^s f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta.$$

where the symbol inside the bracket satisfies the Coifman-Meyer condition given in Theorem A.

Thus it suffices to estimate Π_3 . For simplicity we only consider the term $j = k$. We need to control the following term:

$$\sum_{k \in \mathbf{Z}} \int_{\mathbf{R}^n} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta$$

The inhomogeneous estimate is slightly different because we cannot transfer the derivatives from the product to the high frequency term simply via scaling. More specifically, recall that the key step in the homogeneous estimate was the identity

$$|\xi + \eta|^s = 2^{ks} |2^{-k}(\xi + \eta)|^s |2^{-k}\eta|^{-s} |2^{-k}\eta|^s = |2^{-k}(\xi + \eta)|^s |2^{-k}\eta|^{-s} |\eta|^s.$$

When repeated for this setting, we obtain

$$(1 + |\xi + \eta|^2)^{\frac{s}{2}} = (2^{-2k} + |2^k(\xi + \eta)|^2)^{\frac{s}{2}} (2^{-2k} + |2^k\eta|^2)^{-\frac{s}{2}} (1 + |\eta|^2)^{\frac{s}{2}}.$$

Using (17), we can control these terms when $k \geq 0$, but the constant 2^{-2k} grows unboundedly when $k < 0$. Thus, we need to separate these cases.

On the other hand, when $k < 0$, we note that the term $(1 + |\eta|^2)^{-\frac{s}{2}}$ remains bounded when $\eta \sim 2^k$. This advantage will enable us to handle this case.

We split into the following cases as in Section 3.

Case 1: $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$ or $\frac{1}{2} \leq r < 1$, $1 \leq p, q < \infty$.

As before, we will only show the estimate for the first case, whereas the estimates involving weak L^r norms will immediately follow when the corresponding norms are replaced in the proof below.

First consider the sum when $k \geq 0$.

$$\begin{aligned}
& \Pi_3^1[f, g](x) \\
& := \sum_{k \geq 0} \int_{\mathbf{R}^n} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
& = \sum_{k \geq 0} \int_{\mathbf{R}^n} 2^{ks} (2^{-2k} + |2^{-k}(\xi + \eta)|^2)^{\frac{s}{2}} \Phi(2^{-k-2}(\xi + \eta)) \\
& \quad \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
& = \sum_{k \geq 0} \int_{\mathbf{R}^n} \Phi_s^k(2^{-k}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) \\
& \quad (2^{-2k} + |2^{-k}\eta|^2)^{-\frac{s}{2}} \Psi(2^{-k}\eta) (1 + |\eta|^2)^{\frac{s}{2}} \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta,
\end{aligned}$$

where $\Phi_s^k(\cdot) := (2^{-2k} + |\cdot|^2)^{s/2} \Phi(2^{-2}\cdot)$. Since Ψ is supported on an annulus, pick $\widetilde{\Psi} \in \mathcal{S}(\mathbf{R}^n)$ equal to one on the support of Ψ and supported in the slightly larger annulus $\frac{1}{2} - \frac{1}{10} \leq |\xi| \leq 2 + \frac{1}{5}$. Writing $\Psi = \widetilde{\Psi}\Psi$, we obtain that

$$\begin{aligned}
& \Pi_3^1[f, g](x) \\
& = \sum_{k \geq 0} \int_{\mathbf{R}^n} \Phi_s^k(2^{-k}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) \\
& \quad (2^{-2k} + |2^{-k}\eta|^2)^{-\frac{s}{2}} \widetilde{\Psi}(2^{-k}\eta) \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\
& = \sum_{k \geq 0} \int_{\mathbf{R}^n} \Phi_s^k(2^{-k}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) \widetilde{\Psi}_{-s}^k(2^{-k}\eta) \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta,
\end{aligned}$$

where $\widetilde{\Psi}_{-s}^k(\cdot) := (2^{-2k} + |\cdot|^2)^{s/2} \widetilde{\Psi}(\cdot)$. Now we scale back and expand Φ_s^k and $\widetilde{\Psi}_{-s}^k$ in Fourier series, as in Section 3, over the cube $[-8, 8]^n$. Let $c_{m,k}^s$ and $\widetilde{c}_{l,k}^{-s}$ be the Fourier coefficients of the expansion defined as

$$\begin{aligned}
c_{m,k}^s & := \frac{1}{16^n} \int_{[-8,8]^n} (2^{-2k} + |\xi|^2)^{s/2} \Phi(\xi) e^{-\frac{2\pi i}{16} \langle \xi, m \rangle} d\xi \\
\widetilde{c}_{l,k}^{-s} & := \frac{1}{16^n} \int_{[-8,8]^n} (2^{-2k} + |\xi|^2)^{s/2} \widetilde{\Psi}(\xi) e^{-\frac{2\pi i}{16} \langle \xi, l \rangle} d\xi.
\end{aligned}$$

Then

$$\begin{aligned}
& \Pi_3^1[f, g](x) \\
& = \sum_{k \geq 0} 2^{2kn} \int_{\mathbf{R}^n} \Phi_s^k(\xi + \eta) \Psi(\xi) \widehat{f}(2^k\xi) \widetilde{\Psi}_{-s}^k(\eta) \Psi(\eta) \widehat{J^s g}(2^k\eta) e^{2\pi i 2^k \langle \xi + \eta, x \rangle} d\xi d\eta \\
& = \sum_{k \geq 0} 2^{2kn} \int_{\mathbf{R}^n} \left(\sum_{m \in \mathbf{Z}^n} c_{m,k}^s e^{\frac{2\pi i}{16} \langle \xi + \eta, m \rangle} \right) \chi_{[-8,8]^n}(\xi + \eta) \Psi(\xi) \widehat{f}(2^k\xi)
\end{aligned}$$

$$\begin{aligned}
& \left(\sum_{l \in \mathbf{Z}^n} \tilde{c}_{l,k}^{-s} e^{\frac{2\pi i}{16} \langle \eta, l \rangle} \right) \chi_{[-8,8]^n}(\eta) \Psi(\eta) \widehat{J^s g}(2^k \eta) e^{2\pi i 2^k \langle \xi + \eta, x \rangle} d\xi d\eta \\
&= \sum_{k \geq 0} 2^{2kn} \int_{\mathbf{R}^n} \left(\sum_{m \in \mathbf{Z}^n} c_{m,k}^s e^{\frac{2\pi i}{16} \langle \xi + \eta, m \rangle} \right) \Psi(\xi) \widehat{f}(2^k \xi) \\
& \quad \left(\sum_{l \in \mathbf{Z}^n} \tilde{c}_{l,k}^{-s} e^{\frac{2\pi i}{16} \langle \eta, l \rangle} \right) \Psi(\eta) \widehat{J^s g}(2^k \eta) e^{2\pi i 2^k \langle \xi + \eta, x \rangle} d\xi d\eta,
\end{aligned}$$

since the characteristic functions are equal to one on the support of $\Psi(\xi)\Psi(\eta)$. Lemma 2 implies that $b_m^s := \sup_{k \geq 0} |c_{m,k}^s| = O(|m|^{-n-s})$. The following lemma shows that $\tilde{b}_m^{-s} := \sup_{k \geq 0} |\tilde{c}_{m,k}^{-s}|$ also has a fast decay.

Lemma 3. *Let $\{f_\delta\}_{\delta \in [0,1]}$ be a family of Schwartz functions such that \widehat{f}_δ is supported on a compact set $K \in \mathbf{R}^n$ for all $\delta \in [0, 1]$. If $\sup_{\delta \in [0,1]} \left\| \Delta^N \widehat{f}_\delta \right\|_{L_\xi^\infty} < B$, then there is some constant $C(N)$ such that*

$$\sup_{\delta \in [0,1]} |f_\delta|(x) \leq B|K|C(N)|x|^{-2N}.$$

Remark: Let \mathcal{F}^{-1} denote the inverse Fourier transform, i.e., the Fourier transform composed with the reflection $x \mapsto -x$. Consider the family $\{f_\delta\}_{\delta \in [0,1]}$ defined by $f_\delta := (\delta - \Delta)^{-\frac{s}{2}} \mathcal{F}^{-1}[\tilde{\Psi}]$. Note that $\widehat{f}_\delta(\xi) = (\delta + |\xi|^2)^{-\frac{s}{2}} \tilde{\Psi}(\xi)$ is smooth for $(\delta, \xi) \in [0, 1] \times \mathbf{R}^n$ and compactly supported in ξ . Thus for any $\alpha \in \mathbf{Z}^n$, $\partial^\alpha \widehat{f}_\delta$ is continuous and compactly supported, thus satisfying the condition of the Lemma above. Additionally, f_δ is uniformly bounded for $(\delta, x) \in [0, 1] \times \mathbf{R}^n$ as seen by the following.

$$\left\| (\delta - \Delta)^{-\frac{s}{2}} \mathcal{F}^{-1}[\tilde{\Psi}] \right\|_{L_x^\infty} \leq \left\| (\delta + |\cdot|^2)^{-\frac{s}{2}} \tilde{\Psi} \right\|_{L_\xi^1} \leq \left\| |\cdot|^{-\frac{s}{2}} \tilde{\Psi} \right\|_{L_\xi^1}$$

Thus, we obtain that $\tilde{b}_j^{-s} \leq \sup_{\delta \in [0,1]} |f_\delta|(j) = O((1 + |j|)^{-N})$ for any $N \in \mathbf{N}$.

Proof. Using the identity $\Delta^N e^{i\langle \xi, x \rangle} = C_N |x|^{2N} e^{i\langle \xi, x \rangle}$, we apply Green's theorem:

$$\begin{aligned}
|x|^{2N} |f_\delta|(x) &= \left| \int_{\mathbf{R}^n} \widehat{f}_\delta(\xi) |x|^{2N} e^{2\pi i \langle \xi, x \rangle} d\xi \right| = C_N \left| \int_{\mathbf{R}^n} \widehat{f}_\delta(\xi) \Delta^N e^{2\pi i \langle \xi, x \rangle} d\xi \right| \\
&= C_N \left| \int_K \Delta^N \widehat{f}_\delta(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi \right| \leq C_N |K| \sup_{\delta \in [0,1]} \left\| \Delta^N \widehat{f}_\delta \right\|_{L_\xi^\infty}.
\end{aligned}$$

This proves Lemma 3. □

We now continue the proof of Theorem 1 for the inhomogeneous case. We have

$$\begin{aligned}
& |\Pi_3^1[f, g]|(x) \\
&= \left| \sum_{m, l \in \mathbf{Z}^n} \sum_{k \geq 0} c_{m,k}^s \tilde{c}_{l,k}^{-s} \right.
\end{aligned}$$

$$\begin{aligned}
& \left| \int_{\mathbf{R}^n} e^{2\pi i \langle 2^{-k}\xi, \frac{m}{16} \rangle} \Psi(2^{-k}\xi) \widehat{f}(\xi) e^{2\pi i \langle 2^{-k}\eta, \frac{m+l}{16} \rangle} \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta \right| \\
& \leq \sum_{m,l \in \mathbf{Z}^n} b_k^s \widetilde{b}_k^{-s} \\
& \quad \sum_{k \geq 0} \left| \int_{\mathbf{R}^n} e^{2\pi i \langle 2^{-k}\xi, \frac{m}{16} \rangle} \Psi(2^{-k}\xi) \widehat{f}(\xi) e^{2\pi i \langle 2^{-k}\eta, \frac{m+l}{16} \rangle} \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta \right| \\
& = \sum_{m,l \in \mathbf{Z}^n} b_k^s \widetilde{b}_k^{-s} \sum_{k \geq 0} |[\Delta_k^m f](x) [\Delta_k^{m+l} J^s g](x)|,
\end{aligned}$$

where $[\Delta_k^m f](\cdot) := \int_{\mathbf{R}^n} 2^{kn} \Psi(2^k(\cdot - y) + \frac{m}{16}) f(y) dy$.

Letting $r_* := \min(r, 1)$,

$$\begin{aligned}
\|\Pi_3^1[f, g]\|_{L^r}^{r_*} & \leq \sum_{m,l \in \mathbf{Z}^n} |b_m^s \widetilde{b}_l^{-s}|^{r_*} \left\| \sqrt{\sum_{k \geq 0} |\Delta_k^m f|^2(x)} \sqrt{\sum_{k \geq 0} |\Delta_k^{m+l} J^s g|^2(x)} \right\|_{L^r}^{r_*} \\
& = \sum_{m,l \in \mathbf{Z}^n} |b_m^s \widetilde{b}_l^{-s}|^{r_*} \left\| \sqrt{\sum_{k \geq 0} |\Delta_k^m f|^2} \right\|_{L^p(\mathbf{R}^n)}^{r_*} \left\| \sqrt{\sum_{k \geq 0} |\Delta_k^{m+l} J^s g|^2} \right\|_{L^q(\mathbf{R}^n)}^{r_*}
\end{aligned}$$

for any $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Applying Corollary 1 to the right hand side above gives the estimate for the sum $k \geq 0$, since, as observed, $b_m^s = O(|m|^{-n-s})$ and the series in m and l converge when $s > n/r - n$.

For $k < 0$ define the Fourier coefficient

$$a_m^s := \frac{1}{16^n} \int_{[-8,8]^n} (1 + |\xi|^2)^{s/2} \Phi(2\xi) e^{-\frac{2\pi i}{16} \langle \xi, m \rangle} d\xi.$$

of the function $\Phi(2\xi)$, which is $O((1 + |m|)^{-N})$ for all $N > 0$. Then we have

$$\begin{aligned}
& \Pi_3^2[f, g](x) \\
& := \sum_{k < 0} \int_{\mathbf{R}^n} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Psi(2^{-k}\xi) \widehat{f}(\xi) \Psi(2^{-k}\eta) \widehat{g}(\eta) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta \\
& = \sum_{k < 0} \int_{\mathbf{R}^n} (1 + |\xi + \eta|^2)^{\frac{s}{2}} \Phi(2(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) \\
& \quad (1 + |\eta|^2)^{-\frac{s}{2}} \Phi(2\eta) \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta \\
& = \sum_{k < 0} \int_{\mathbf{R}^n} \left(\sum_{m \in \mathbf{Z}^n} a_m^s e^{\frac{2\pi i}{16} \langle \xi+\eta, m \rangle} \right) \chi_{[-8,8]^n}(\xi + \eta) \Psi(2^{-k}\xi) \widehat{f}(\xi) \\
& \quad \left(\sum_{l \in \mathbf{Z}^n} a_l^{-s} e^{\frac{2\pi i}{16} \langle \eta, l \rangle} \right) \chi_{[-8,8]^n}(\eta) \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi+\eta, x \rangle} d\xi d\eta \\
& = \sum_{k < 0} \int_{\mathbf{R}^n} \left(\sum_{m \in \mathbf{Z}^n} a_m^s e^{\frac{2\pi i}{16} \langle \xi+\eta, m \rangle} \right) \Psi(2^{-k}\xi) \widehat{f}(\xi)
\end{aligned}$$

$$\left(\sum_{l \in \mathbf{Z}^n} a_l^{-s} e^{\frac{2\pi i}{16} \langle \eta, l \rangle} \right) \Psi(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta,$$

since the characteristic functions are equal to 1 on the support of $\Psi(2^{-k}\xi)\Psi(2^{-k}\eta)$, since $k < 0$. Note that c_m^s and c_m^{-s} are $O(|m|^{-N})$ for any $N \in \mathbf{N}$. We conclude that

$$\begin{aligned} & \Pi_3^2[f, g](x) \\ &= \sum_{m, l \in \mathbf{Z}^n} a_m^s a_l^{-s} \\ & \quad \sum_{k < 0} \int_{\mathbf{R}^n} \Psi(2^{-k}\xi) e^{2\pi i \langle \xi, \frac{m}{16} \rangle} \widehat{f}(\xi) \Psi(2^{-k}\eta) e^{2\pi i \langle \eta, \frac{m+l}{16} \rangle} \widehat{J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \sum_{m, l \in \mathbf{Z}^n} a_m^s a_l^{-s} \sum_{k < 0} \int_{\mathbf{R}^n} \Psi(2^{-k}\xi) \widehat{\tau_{\frac{m}{16}} f}(\xi) \Psi(2^{-k}\eta) \widehat{\tau_{\frac{m+l}{16}} J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \sum_{m, l \in \mathbf{Z}^n} a_m^s a_l^{-s} \sum_{k < 0} [\Delta_k \tau_{\frac{m}{16}} f](x) [\Delta_k \tau_{\frac{m+l}{16}} J^s g](x), \end{aligned}$$

where τ_m is the translation operator $[\tau_m f](x) = f(x - m)$. Taking the L^r norm, we obtain

$$\begin{aligned} \|\Pi_3^2[f, g]\|_{L^r}^{r^*} &\leq \sum_{m, l \in \mathbf{Z}^n} |a_m^s a_l^{-s}|^{r^*} \left\| \sqrt{\sum_{k \in \mathbf{Z}} |\Delta_k \tau_m f|^2} \right\|_{L^p}^{r^*} \left\| \sqrt{\sum_{k \in \mathbf{Z}} |\Delta_k \tau_{m+l} J^s g|^2} \right\|_{L^q}^{r^*} \\ &\leq \sum_{m, l \in \mathbf{Z}^n} |a_m^s a_l^{-s}|^{r^*} \|f\|_{L^p}^{r^*} \|J^s g\|_{L^q}^{r^*}. \end{aligned}$$

In view of the rapid decay of a_m^s and a_m^{-s} , we conclude the proof of Case 1.

Case 2: $1 < r < \infty$, $(p, q) \in \{(r, \infty), (\infty, r)\}$

We again adapt the proof given in [1]. Following the computations in Section 3, for $j \geq 2$, $[\Delta_j \Pi_3^1[f, g]](x)$ can be written as

$$\begin{aligned} & \sum_{k \geq j-2} \int_{\mathbf{R}^{2n}} 2^{js} \Psi_{j,s}(2^{-j}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) 2^{-ks} \Psi_{k,-s}(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ & \leq 2^{js} \left(\sum_{k \geq j-2} 2^{-2ks} \right)^{\frac{1}{2}} \left(\sum_{k \geq j-2} \left| \Delta_{j,s} \left[[\Delta_k f] [\Delta_{k,-s} J^s g] \right](x) \right|^2 \right)^{\frac{1}{2}} \end{aligned}$$

where $\Psi_{j,s}(\cdot) := (2^{-2j} + |\cdot|^2)^{s/2} \Psi(\cdot)$; and the operators $\mathcal{F}[\Delta_{j,s} f] := \Psi_{j,s}(2^{-j}\cdot) \widehat{f}(\cdot)$. Note that the family $\{\Delta_{j,s}\}_{j \geq 0}$ is not a Littlewood-Paley family in the usual sense, i.e. it is not given by convolution with L^1 dilations of a single kernel. Rather, it is given by convolution with kernels that are different for each $j \geq 0$. Below, we will show that $\{\Delta_{j,s}\}_{j \geq 0} : L^p \rightarrow L^p \ell^2$ for $1 < p < \infty$ and $s \in \mathbf{R}$.

When $j < 2$, $[\Delta_j \Pi_3^1[f, g]](x)$ can be written as

$$\left| \sum_{k \geq 0} \int_{\mathbf{R}^{2n}} \Phi_s(\xi + \eta) \Psi(2^{-j}(\xi + \eta)) \Psi(2^{-k}\xi) \widehat{f}(\xi) 2^{-ks} \Psi_{k,-s}(2^{-k}\eta) \widehat{J^s g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \right|$$

$$\leq \left(\sum_{k \geq 0} 2^{-2ks} \right)^{\frac{1}{2}} \left(\sum_{k \geq 0} |S_2^s \Delta_j [[\Delta_k f] [\Delta_{k,-s} J^s g]](x)|^2 \right)^{\frac{1}{2}},$$

where $\Phi_s(\cdot) := (1 + |\cdot|^2)^{s/2} \Phi(2^2 \cdot)$ and $S_2^s f = \mathcal{F}^{-1}[\widehat{\Phi_s f}]$. Then,

$$\begin{aligned} \|\Pi_3^1[f, g]\|_{L^r} &\leq C(r, n, s) \left(\left\| \left(\sum_{j \geq 2} \sum_{k \geq j-2} |\Delta_{j,s} [\Delta_k f \Delta_{k,-s} J^s g]|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \right. \\ &\quad \left. + \left\| \left(\sum_{j < 2} \sum_{k \geq 0} |S_2^s \Delta_j [\Delta_k f \Delta_{k,-s} J^s g]|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \right). \end{aligned}$$

The operator $\{S_2^s \Delta_j\}_{j \in \mathbf{Z}} = \{\Delta_j S_2^s\}_{j \in \mathbf{Z}} : L^r \rightarrow L^r \ell^2$ is clearly bounded for $1 < r < \infty$. Next, we will show that $\{\Delta_{j,s}\}_{j \geq 0} : L^r \rightarrow L^r \ell^2$. Recall that we have introduced above $\widetilde{\Psi} \in \mathcal{S}(\mathbf{R}^n)$ supported on an slightly larger annulus than that of Ψ such that $\Psi = \widetilde{\Psi} \Psi$. We write

$$[\widetilde{\Delta}_j u](x) = \int_{\mathbf{R}^n} \widetilde{\Psi}_j^s(2^{-j} \xi) \Psi(2^{-j} \xi) \widehat{u}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi$$

where $\widetilde{\Psi}_j^s(\cdot) := (2^{-2j} + |2^{-j} \cdot|^2)^{\frac{s}{2}} \widetilde{\Psi}(\cdot)$. Expanding $\widetilde{\Psi}_j^s$ in Fourier series with coefficients denoted $\widetilde{c}_{m,j}^s$, we can write $\widetilde{\Delta}_j u = \sum_{m \in \mathbf{Z}^n} \widetilde{c}_{m,j}^s \Delta_j^m u$ where

$$[\Delta_j^m u](x) = \int_{\mathbf{R}^n} e^{\frac{2\pi i}{16} \langle 2^{-j} \xi, m \rangle} \Psi(2^{-j} \xi) \widehat{u}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

Defining $\widetilde{b}_m^s := \sup_{j \geq 0} |\widetilde{c}_{m,j}^s|$, we recall from Lemma 3 that $\widetilde{b}_m^s = O((1 + |m|)^{-N})$ for any $N \in \mathbf{N}$. Thus, applying Corollary 1,

$$\begin{aligned} \left\| \left(\sum_{j \geq 0} |\widetilde{\Delta}_j u|^2 \right)^{\frac{1}{2}} \right\|_{L^r} &\leq \left\| \left(\sum_{j \geq 0} \left| \sum_{m \in \mathbf{Z}^n} \widetilde{b}_m^s |\Delta_j^m u| \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \leq \sum_{m \in \mathbf{Z}^n} \widetilde{b}_m^s \left\| \left(\sum_{j \geq 0} |\Delta_j^m u|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \\ &\leq C(n, r) \sum_{m \in \mathbf{Z}^n} \widetilde{b}_m^s \ln(1 + |m|) \|u\|_{L^r}. \end{aligned}$$

Using [10, Proposition 4.6.4], we extend the operators $\{S_2^s \Delta_j\}_{j \in \mathbf{Z}}$ and $\{\Delta_{j,s}\}_{j \geq 0}$ from $L^r \rightarrow L^r \ell^2$ to $L^r \ell^2 \rightarrow L^r \ell^2 \ell^2$ for $1 < r < \infty$ and obtain

$$\begin{aligned} \|\Pi_3^1[f, g]\|_{L^r} &\leq C(n, r, s) \left\| \left(\sum_{k \geq 0} |\Delta_k f \Delta_{k,-s} J^s g|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \\ &\leq C(n, r, s) \sup_{k \geq 0} \|\Delta_{k,-s} J^s g\|_{L^\infty} \|f\|_{L^r} \\ &\leq \|f\|_{L^r} \|J^s g\|_{L^\infty} \sup_{k \geq 0} \left\| \widehat{\Psi_k^{-s}} \right\|_{L^1} \end{aligned}$$

for $1 < r < \infty$. Applying Lemma 3 and the remark following, we obtain that

$$\sup_{k \geq 0} \left| \widehat{\Psi_k^{-s}} \right| (x) = \sup_{\geq 0} \left| \mathcal{F} \left[(2^{-2k} - |\cdot|^2)^{-s/2} \widetilde{\Psi}(\cdot) \right] \right| (x) = O((1 + |x|)^{-N})$$

for any $N \in \mathbf{N}$. Taking $N > n$ gives the necessary estimate for $\Pi_3^1[f, g]$.

It remains to obtain the endpoint estimates for $\Pi_3^2[f, g]$. For this, we write

$$\Pi_3^2[f, g] = S_2^s \left[\sum_{k \leq 0} (\Delta_k f) (\Delta_k S_2^{-s} J^s g) \right].$$

Note that, for any $s \in \mathbf{R}$, S_2^s is a L^p multiplier for $1 \leq p \leq \infty$ since it is a convolution with an $L^1(\mathbf{R}^n)$ function. Also, the symbol $\sum_{k < 0} \Psi(2^{-k}\xi)\Psi(2^{-k}\eta)$ satisfies the Coifman-Meyer condition in Theorem A. Thus we obtain

$$\left\| \Pi_3^2[f, g] \right\|_{L^r} \leq C(n, r, s) \|f\|_{L^p} \left\| S_2^{-s} J^s g \right\|_{L^q} \leq C(n, r, s) \|f\|_{L^p} \|J^s g\|_{L^q}$$

for any $1 \leq r < \infty$, $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. \square

5. MULTI-PARAMETER KATO-PONCE INEQUALITY

Let $f, g \in \mathcal{S}(\mathbf{R}^n)$, we want to prove the multi-parameter Kato-Ponce inequality. Write $\mathbf{R}^n = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \cdots \times \mathbf{R}^{n_d}$ and denote for $x \in \mathbf{R}^n$, $x = (x_1, x_2, \dots, x_d)$ where $x_j \in \mathbf{R}^{n_j}$ for $j = 1, 2, \dots, d$.

For $s \in \mathbf{R}$, define fractional partial derivatives $|D_{x_j}|^s$ by $\mathcal{F}[|D_{x_j}|^s f](\xi) := |\xi_j|^s \widehat{f}(\xi)$. Let $E := \{1, 2, \dots, d\}$ and $\mathcal{P}[E]$ be its power set. For $B \in \mathcal{P}[E]$, denote $D_{x(B)}^{s(B)} := \prod_{j \in B} D_{x_j}^{s_j}$. Then the multi-parameter homogeneous Kato-Ponce inequality can be stated as follows:

Theorem 4. *Given $s_j > \max(n_j/r - n_j, 0)$ for $j = 1, 2, \dots, d$ there exists $C = C(d, n_j, r, s_j, p(B), q(B))$ so that for all $f, g \in \mathcal{S}(\mathbf{R}^n)$,*

$$(21) \quad \left\| D_{x(E)}^{s(E)}[fg] \right\|_{L^r(\mathbf{R}^n)} \leq C \sum_{B \in \mathcal{P}[E]} \left\| D_{x(B)}^{s(B)} f \right\|_{L^{p(B)}(\mathbf{R}^n)} \left\| D_{x(E \setminus B)}^{s(E \setminus B)} g \right\|_{L^{q(B)}(\mathbf{R}^n)}$$

for $\frac{1}{2} < r < \infty$, $1 < p(B), q(B) \leq \infty$ satisfying $\frac{1}{p(B)} + \frac{1}{q(B)} = \frac{1}{r}$.

Theorem 5. *When $s_j \leq \max(n_j/r - n_j, 0)$ for some $j = 1, 2, \dots, d$, (21) fails.*

Proof of Theorem 5. Let $f^{(n_j)} \in \mathcal{S}(\mathbf{R}^{n_j})$ be non-zero functions for $j = 1, \dots, d$. Then define $F(x) := \prod_{j=1}^d F^{(n_j)}(x_j)$. Then $\widehat{F}(\xi) = \prod_{j=1}^d \widehat{f^{(n_j)}}(\xi_j)$ and $[D_{x(E)}^{s(E)} F](x) = \prod_{j=1}^d [D_{x_j}^{s_j} f^{(n_j)}](x_j)$. Thus their $L^p(\mathbf{R}^n)$ norms split into a product of $L^p(\mathbf{R}^{n_j})$ norms. Thus letting $f = F$ and $g = \overline{F}$ in (21), Lemma 1 gives that the left hand side is infinite if $s \leq \frac{n_j}{r} - n_j$ for any $j = 1, \dots, d$, while the right hand side is finite as long as $p(B), q(B) > 1$.

The argument for $s_j < 0$ easily follows by a similar argument as in Section 3. \square

Next we will prove Theorem 4. First we make a few remarks below.

- In the case of \mathbf{R}^2 , Theorem 4 stated in the appendix of [17]. However, in view of Theorem 5, we note that the inequality [17, Equation (61)] holds only when $\min(\alpha, \beta) > \frac{1}{r} - 1$ for $r < 1$. *This point has been corrected in [19].*
- The weak L^r endpoints for these estimates could be false due to the fact that $L^{r,\infty}$ norms cannot be iterated, i.e. $\|f\|_{L^{r,\infty}(\mathbf{R}^2)} \neq \left\| \|f\|_{L^{r,\infty}(\mathbf{R})} \right\|_{L^{r,\infty}(\mathbf{R})}$.
- From the proof given in Section 4 and the proof to be presented below, it will be apparent that the operators $D_{x_j}^s$ can be replaced by $J_{x_j}^s$ defined similarly. We have not included this generalization in order to simplify the argument.

The proof of Theorem 4 is an iteration of the proof of Theorem 1 in Section 3, using multi-parameter Littlewood-Paley decompositions. We introduce the corresponding operators here.

Let $\Phi^{(j)} \in \mathcal{S}(\mathbf{R}^{n_j})$ be such that $\Phi \equiv 1$ when $|\xi| \leq 1$ and is supported on $|\xi| \leq 2$. Define $\Psi^{(j)}(\cdot) := \Phi^{(j)}(\cdot) - \Phi^{(j)}(2\cdot)$. For technical reasons, we also define $\Psi^{[j]} \in \mathcal{S}(\mathbf{R}^{n_j})$ to be supported on an annulus, and satisfying $\sum_{k \in \mathbf{Z}} |\Psi^{[j]}(2^{-k}\xi_j)|^2 = 1$ for all $\xi_j \in \mathbf{R}^{n_j} \setminus \{0\}$.

Define the operator $S_k^{(j)}$ by $\mathcal{F}[S_k^{(j)} f](\xi) = \Phi^{(j)}(2^{-k}\xi_j)\widehat{f}(\xi)$; $\Delta_k^{(j)}$ by $\mathcal{F}[\Delta_k^{(j)} f](\xi) = \Psi^{(j)}(2^{-k}\xi_j)\widehat{f}(\xi)$; and $\Delta_k^{[j]}$ by $\mathcal{F}[\Delta_k^{[j]} f](\xi) = \Psi^{[j]}(2^{-k}\xi_j)\widehat{f}(\xi)$. Given $B \subset \{1, 2, \dots, d\}$, define $S_{k(B)}^{(B)}$, $\Delta_{k(B)}^{(B)}$, $\Delta_{k(B)}^{[B]}$ by $\prod_{j \in B} S_{k_j}^{(j)}$, $\prod_{j \in B} \Delta_{k_j}^{(j)}$, $\prod_{j \in B} \Delta_{k_j}^{[j]}$ respectively.

The following lemma shows the boundedness of the corresponding square-functions in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. This is a slight generalization of [10, Theorem 5.1.6].

Lemma 4. *Let $\widetilde{\Psi}^{(j)} \in \mathcal{S}(\mathbf{R}^{n_j})$ satisfy the conditions (4) and (5) in Theorem 3 with the constant B_j^2 , B^2 respectively for $j = 1, 2, \dots, d$. Let define $\widetilde{\Delta}_k^{(j)}$ by $\mathcal{F} \left[\widetilde{\Delta}_k^{(j)} u \right](\xi) = \widetilde{\Psi}^{(n_j)}(2^{-k}\xi_j)\widehat{u}(\xi)$. Then for all $u \in \mathcal{S}(\mathbf{R}^n)$, there exists $C_j = C(n_j, p) < \infty$ satisfying*

$$\left\| \sqrt{\sum_{k_1, \dots, k_d \in \mathbf{Z}} \left| \widetilde{\Delta}_{k_1}^{(1)} \cdots \widetilde{\Delta}_{k_d}^{(d)} u \right|^2} \right\|_{L^p(\mathbf{R}^n)} \leq \left[\prod_{j=1}^d C_j B_j \max(p, (p-1)^{-1}) \right] \|u\|_{L^p(\mathbf{R}^n)}.$$

The proof of the lemma above is simply an iteration of Theorem 3 d times whilst commuting the $L^p(\mathbf{R}^{n_j})$ norms. We refer to the proof given in [10, Theorem 5.1.6] for this calculations.

The following lemma is due to Ruan, [21, Theorem 3.2].

Lemma 5. *Let $0 < p < \infty$. For all $u \in \mathcal{S}(\mathbf{R}^n)$, there exists $C = C(n, p)$ satisfying*

$$\|u\|_{L^p(\mathbf{R}^n)} \leq C \left\| \sqrt{\sum_{k_1, \dots, k_d \in \mathbf{Z}} \left| \Delta_{k_1}^{[1]} \cdots \Delta_{k_d}^{[d]} u \right|^2} \right\|_{L^p(\mathbf{R}^n)}.$$

For $1 < p < \infty$, this immediately follows from duality and Lemma 4. However, for $0 < p \leq 1$, this is a consequence of a multi-parameter square-function characterization of Hardy spaces $H^p(\mathbf{R}^n)$. We refer to [21] for details.

Proof of Theorem 4. We introduce notation to aid the computations. Although we strive to use clear and accurate notation throughout, it will be inevitable at times to be flexible for the sake of exposition. We have

$$\begin{aligned} \left[D_{x(E)}^{s(E)}[fg] \right] (x) &= \iint_{\mathbf{R}^{2n}} \left[\prod_{j=1}^d |\xi_j + \eta_j|^{s_j} \right] \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta \\ &= \iint_{\mathbf{R}^{2n}} \prod_{j=1}^d \left[|\xi_j + \eta_j|^{s_j} \sum_{k,l \in \mathbf{Z}} \Psi^{(j)}(2^{-k} \xi_j) \Psi^{(j)}(2^{-l} \eta_j) \right] \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta. \end{aligned}$$

For each $j = 1, 2, \dots, d$, split the sum inside the square bracket into $\sum_{l < k-1} \cdot + \sum_{k < l-1} \cdot + \sum_{|k-l| \leq 1} \cdot =: M_j^1 + M_j^2 + M_j^3$ and take the product over $j = 1, 2, \dots, d$, to obtain

$$\prod_{j=1}^d [M_j^1 + M_j^2 + M_j^3](\xi_j, \eta_j) = \sum_{a_1=1}^3 \sum_{a_2=1}^3 \cdots \sum_{a_d=1}^3 \left[\prod_{j=1}^d M_j^{a_j}(\xi_j, \eta_j) \right].$$

Define $J := [\mathbf{Z}/3\mathbf{Z}]^d$ to be the set of n -tuples where each component is from $\{1, 2, 3\}$. Given $A = (a_1, a_2, \dots, a_n) \in J$, define $M_A(\xi, \eta) := \prod_{j=1}^d M_j^{a_j}(\xi_j, \eta_j)$ so that the sum above can be expressed as $\sum_{A \in J} M_A(\xi, \eta)$. Denoting

$$\Pi_A[f, g](x) := \int_{\mathbf{R}^{2n}} M_A(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta,$$

we have $D_{x(E)}^{s(E)}[fg] = \sum_{A \in J} \Pi_A[f, g]$. Thus, given $A \in J$, it suffices to show (21) for $\Pi_A[f, g]$.

Fix $A = (a_1, a_2, \dots, a_d) \in J$. We define the sets A_1, A_2, A_3 such that for $\alpha = 1, 2, 3$, $A_\alpha := \{j \in E : a_j = \alpha\}$. For any $A \in J$, $\{A_1, A_2, A_3\}$ forms a partition of E . For $\alpha, \beta \in \{1, 2, 3\}$, $A_{\alpha, \beta} := A_\alpha \cup A_\beta$. Roughly speaking, $A_{1,2}$ represents the components which have the high-low frequency interactions, and A_3 represents the ones with high-high interactions.

Iterate the $L^r(\mathbf{R}^n)$ norm by $\|\Pi_A[f, g]\|_{L^r(\mathbf{R}^n)} = \left\| \left\| \Pi_A[f, g] \right\|_{L_{A_{1,2}}^r(\mathbf{R}^{|A_{1,2}|})} \right\|_{L_{A_3}^r(\mathbf{R}^{|A_3|})}$.

For the norm inside, we apply Lemma 5 to obtain

$$\|\Pi_A[f, g]\|_{L_{A_{1,2}}^r(\mathbf{R}^{|A_{1,2}|})} \leq C(n, r) \left\| \sqrt{\sum_{m_j \in \mathbf{Z}; j \in A_{1,2}} \left| \Delta_{m(A_{1,2})}^{[A_{1,2}]} \Pi_A[f, g] \right|^2} \right\|_{L_{A_{1,2}}^r(\mathbf{R}^{|A_{1,2}|})}.$$

For $j \in E$ and $\alpha = 1, 2$, denote $\widetilde{M}_{j,m}^\alpha := \psi^{[j]}(2^{-m}(\xi + \eta)) M_j^\alpha(\xi, \eta)$, and $\widetilde{M}_{j,m}^3 := M_j^3$. Then

$$\left[\Delta_{m(A_{1,2})}^{[A_{1,2}]} \Pi_A[f, g] \right] (x) = \int_{\mathbf{R}^{2n}} \left[\prod_{j=1}^d \widetilde{M}_{j,m_j}^{a_j}(\xi_j, \eta_j) \right] \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta.$$

We analyze the symbols $\widetilde{M_{j,m_j}^{a_j}}$ and transfer the fractional derivatives onto the high-frequency term.

$$\begin{aligned} \widetilde{M_{j,m}^1} &= |\xi_j + \eta_j|^{s_j} \Psi^{[j]}(2^{-m}(\xi_j + \eta_j)) \sum_{k \in \mathbf{Z}} \Psi^{(j)}(2^{-k}\xi_j) \Phi^{(j)}(2^{-k+2}\eta_j) \\ &= |\xi_j + \eta_j|^{s_j} \sum_{|k-m| \leq 3} \Psi^{[j]}(2^{-m}(\xi_j + \eta_j)) \Psi^{(j)}(2^{-k}\xi_j) \Phi^{(j)}(2^{-k+2}\eta_j) \\ &= \sum_{|k-m| \leq 3} 2^{(m-k)s} \Psi_{s_j}^{[j]}(2^{-m}(\xi_j + \eta_j)) \Psi_{-s_j}^{(j)}(2^{-k}\xi_j) \Phi(2^{-k+2}\eta_j) |\xi_j|^{s_j}, \end{aligned}$$

where $\Psi_{s_j}^{[j]}(\cdot) := |\cdot|^{s_j} \Psi^{[j]}(\cdot)$ and $\Psi_{-s_j}^{(j)}(\cdot) := |\cdot|^{-s_j} \Psi^{(j)}(\cdot)$. Note that $\Psi_{-s_j}^{(j)} \in \mathcal{S}(\mathbf{R}^{n_j})$, so that the slight change in the operator $\Delta_m^{(j)}$ is not significant. Thus we will ignore this difference. Also, we will replace the finite sum above by a larger constant in the end. Expanding $\Psi_{s_j}^{[j]}$ into its Fourier series, we obtain

$$\begin{aligned} \widetilde{M_{j,m}^1}(\xi_j, \eta_j) &= \sum_{l \in \mathbf{Z}^{n_j}} \widetilde{c_l^{s_j}} e^{\frac{2\pi i}{16} \langle 2^{-k}(\xi_j + \eta_j), l \rangle} \Psi_{-s_j}^{(j)}(2^{-k}\xi_j) \Phi^{(j)}(2^{-k+2}\eta_j) \xi_{[-8,8]^{n_j}}(2^{-k}(\xi + \eta)) |\xi_j|^{s_j} \\ &= \sum_{l \in \mathbf{Z}^{n_j}} \widetilde{c_l^{s_j}} e^{\frac{2\pi i}{16} \langle 2^{-k}\xi_j, l \rangle} \Psi_{-s_j}^{(j)}(2^{-k}\xi_j) e^{\frac{2\pi i}{16} \langle 2^{-k}\eta_j, l \rangle} \Phi^{(j)}(2^{-k+2}\eta_j) |\xi_j|^{s_j} \end{aligned}$$

where $\widetilde{c_l^{s_j}} := 16^{-n} \int_{[-8,8]^{n_j}} |\xi_j|^{s_j} \Psi^{(n_j)}(\xi_j) e^{-\frac{2\pi i}{16} \langle \xi_j, l \rangle} d\xi_j$. Similarly,

$$\widetilde{M_{j,m}^2}(\xi_j, \eta_j) = \sum_{l \in \mathbf{Z}^{n_j}} \widetilde{c_l^{s_j}} e^{\frac{2\pi i}{16} \langle 2^{-k}\xi_j, l \rangle} \Phi^{(j)}(2^{-k+2}\xi_j) e^{\frac{2\pi i}{16} \langle 2^{-k}\eta_j, l \rangle} \Psi_{-s_j}^{(j)}(2^{-k}\eta_j) |\eta_j|^{s_j}.$$

For $\widetilde{M_{j,m}^3} = M_j^3$,

$$M_j^3(\xi_j, \eta_j) = \sum_{l \in \mathbf{Z}^{n_j}} \widetilde{c_l^{s_j}} \sum_{k \in \mathbf{Z}} e^{\frac{2\pi i}{16} \langle 2^{-k}\xi_j, l \rangle} \Psi(2^{-k}\xi_j) e^{\frac{2\pi i}{16} \langle 2^{-k}\eta_j, l \rangle} \Psi_{-s_j}^{(j)}(2^{-k}\eta_j) |\eta_j|^{s_j}$$

where $\widetilde{c_l^{s_j}} := 16^{-n} \int_{[-8,8]^{n_j}} |\xi_j|^{s_j} \Phi^{(j)}(\xi_j) e^{-\frac{2\pi i}{16} \langle \xi_j, l \rangle} d\xi_j$ by Lemma 1. Recall that $\widetilde{c_l^{s_j}}, \widetilde{c_l^{s_j}} = O((1 + |l|)^{-n_j - s_j})$. As in the previous sections, we can pull the summation in l_j for $j = 1, 2, \dots, d$ outside the norm. Also, the shift $e^{\frac{2\pi i}{16} \langle 2^{-k}\xi_j, l \rangle}$ acting on the Littlewood-Paley operators creates a logarithmic term via Lemma 4, which can be controlled due to the fast decay of $|\widetilde{c_l^{s_j}}|^{r^*}$ and $|\widetilde{c_l^{s_j}}|^{r^*}$ where $r_* = \min(r, 1)$. Thus, we ignore the summation in l and the shift operators $e^{\frac{2\pi i}{16} \langle 2^{-k}\cdot, l \rangle}$ acting on $\widetilde{\Delta_k^{(j)}}$.

Therefore, we can reduce the key expression as follows:

$$\Delta_{m(A_{1,2})}^{[A_{1,2}]} \Pi_A[f, g] \approx \sum_{\substack{k_j \in \mathbf{Z}; \\ j \in A_3}} \left[\Delta_{m(A_1)}^{(A_1)} S_{m(A_2)}^{(A_2)} \Delta_{k(A_3)}^{(A_3)} D_{x(A_1)}^{s(A_1)} f \right] \left[S_{k(A_1)}^{(A_1)} \Delta_{m(A_2)}^{(A_2)} \Delta_{k(A_3)}^{(A_3)} D_{x(A_{1,3})}^{s(A_{1,3})} g \right].$$

Now applying the Cauchy-Schwarz inequality for the summation in $k_j : j \in A_3$, and the $\ell^1 - \ell^\infty$ Hölder inequality for $k_j : j \in A_{1,2}$, we obtain

$$\begin{aligned} \|\Pi_A[f, g]\|_{L^r(\mathbf{R}^n)} &\lesssim_{n,r} \left\| \left(\sum_{\substack{m_j \in \mathbf{Z}; \\ j \in A_{1,2}}} \left| \Delta_{m(A_{1,2})}^{[A_{1,2}]} \Pi_A[f, g] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r(\mathbf{R}^n)} \\ &\lesssim \left\| \left(\sum_{\substack{k_j \in \mathbf{Z} \\ j \in A_{1,3}}} \left| M^{A_2} \Delta_{k(A_{1,3})}^{(A_{1,3})} D_{x(A_1)}^{s(A_1)} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \left\| \left(\sum_{\substack{k_j \in \mathbf{Z} \\ j \in A_{2,3}}} \left| M^{A_1} \Delta_{k(A_{2,3})}^{(A_{2,3})} D_{x(A_{2,3})}^{s(A_{2,3})} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $1 < p, q < \infty$, where M^{A_2} and M^{A_1} represents the appropriate Hardy-Littlewood maximal functions. We apply Fefferman-Stein's inequality [9] to remove the maximal functions. Then the quantity above is controlled by

$$\left\| \left(\sum_{\substack{k_j \in \mathbf{Z} \\ j \in A_{1,3}}} \left| \Delta_{k(A_{1,3})}^{(A_{1,3})} D_{x(A_1)}^{s(A_1)} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \left\| \left(\sum_{\substack{k_j \in \mathbf{Z} \\ j \in A_{2,3}}} \left| \Delta_{k(A_{2,3})}^{(A_{2,3})} D_{x(A_{2,3})}^{s(A_{2,3})} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\mathbf{R}^n)}.$$

Commuting the $L^r(\mathbf{R}^{n_j})$ norms appropriately so that we can apply Lemma 4, this quantity is bounded by

$$\left\| D_{x(A_1)}^{s(A_1)} f \right\|_{L^p(\mathbf{R}^n)} \left\| D_{x(A_{2,3})}^{s(A_{2,3})} g \right\|_{L^q(\mathbf{R}^n)}.$$

This concludes the proof in the cases $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$.

Consider the endpoint case $1 < r < \infty$, $p = r$ and $q = \infty$. We begin by applying Lemma 5 and following the computations from Section 3.

$$\begin{aligned} \|\Pi_A[f, g]\|_{L^r(\mathbf{R}^n)} &\lesssim_{n,r} \left\| \sqrt{\sum_{m_j \in \mathbf{Z}; j \in E} \left| \Delta_{m(E)}^{[E]} \Pi_A[f, g] \right|^2} \right\|_{L^r(\mathbf{R}^n)} \\ &\lesssim \left\| \sqrt{\sum_{m_j, k_j \in \mathbf{Z}; j \in E} \left| \widetilde{\Delta_{m(E)}^{[E]}} \left[\Delta_{k(A_{1,3})}^{(A_{1,3})} S_{k(A_2)}^{(A_2)} D_{x(A_1)}^{s(A_1)} f \right] \left[S_{k(A_1)}^{(A_1)} \Delta_{k(A_{2,3})}^{(A_{2,3})} D_{x(A_{2,3})}^{s(A_{2,3})} g \right] \right|^2} \right\|_{L^r(\mathbf{R}^n)} \end{aligned}$$

where $\widetilde{\Delta_{m(E)}^{[E]}}$ is a bounded operator mapping $L^p \rightarrow L^p \ell^2$ due to Lemma 4, so that we can apply [10, Proposition 4.6.4]. Noting that $\sup_{k \in \mathbf{Z}} S_{k(A_1)}^{(A_1)} \Delta_{k(A_{2,3})}^{(A_{2,3})} : L^\infty(\mathbf{R}^n) \rightarrow L^\infty(\mathbf{R}^n)$ is a bounded operator, the endpoint estimates follow as before.

This concludes the proof of Theorem 4. \square

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